Magnetically Coupled Circuits

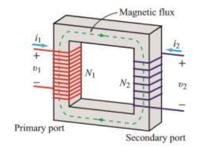
ECE 101F19 Lecture 19, Dec 3, 2019

Contents

Objectives

Learn to:

- Incorporate mutual coupling in magnetically coupled circuits.
- Analyze circuits containing magnetically coupled coils.
- Relate input to output voltages, currents, and impedances for magnetically coupled transformers, including ideal transformers and three-phase transformers.



When two physically unconnected *inductors* are in close proximity to one another, current flow through one of them induces a *magnetically coupled voltage* across the other one. Magnetic coupling may be intentional or not. Highly coupled *voltage transformers* used in power distribution networks are an example of intentional coupling. If the coupling between two coils in a circuit is unintentional but significant, its effects should be incorporated into the analysis of the circuit.

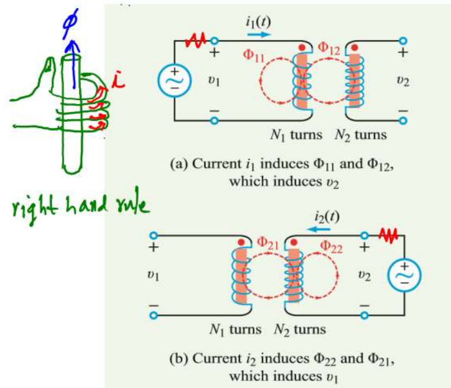
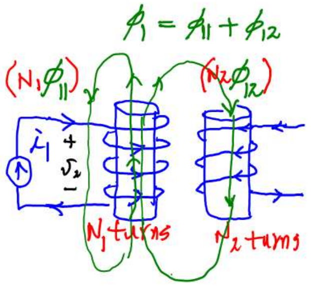


Figure 11-1: Magnetically coupled coils.



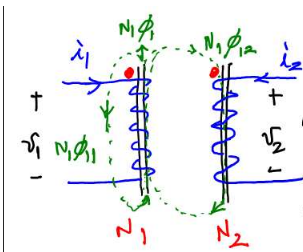
self inductance

$$L_1 = \frac{N_1 \phi_1}{i_1}$$

$$v_1 = \frac{d}{dt}(N_1 \phi_1) = \frac{d}{dt}(L_1 i_1) = L_1 \frac{di_1}{dt}$$

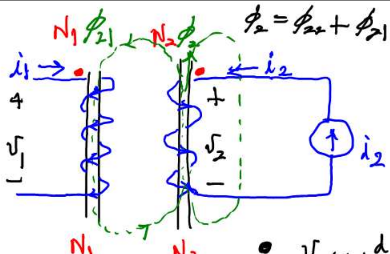
the voltage induced in the secondary coil

$$v_2 = \frac{d}{dt}(N_2 \phi_{12}) = \frac{d}{dt}(+M_{21} i_1) = +M_{21}\frac{di_1}{dt}$$



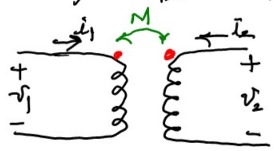
- $v_1 = L_1 \frac{di_1}{dt}$
- $v_2 = +M_{21}\frac{di_1}{dt}$

$\phi_2 = \phi_{22} + \phi_{21}$



- $v_2 = \frac{d}{dt} N_2 \phi_2 = \frac{d}{dt}(L_2 i_2) = L_2 \frac{di_2}{dt}$
- $v_1 = \frac{d}{dt}(N_1 \phi_{21}) = \frac{d}{dt}(+M_{12} i_2) = +M_{12}\frac{di_2}{dt}$

Usually, $M_{12} = M_{21} = M$



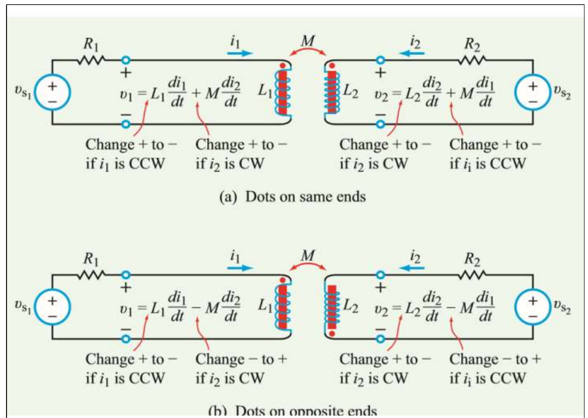
Time Domain $v_1 = L_1 \frac{di_1}{dt} + M\frac{di_2}{dt}$

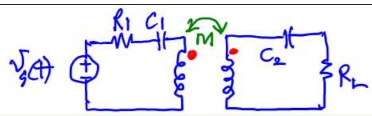
$v_2 = M\frac{di_1}{dt} + L_2\frac{di_2}{dt}$

AC Domain

$\mathbf{V}_1 = j\omega L_1 \mathbf{I}_1 + j\omega M \mathbf{I}_2$

$\mathbf{V}_2 = j\omega M \mathbf{I}_1 + j\omega L_2 \mathbf{I}_2$





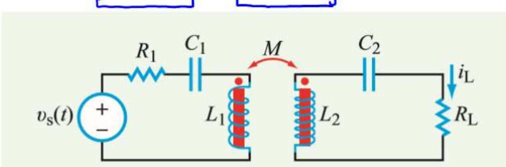
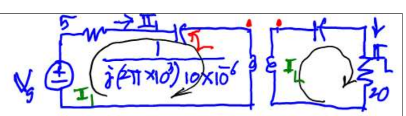


Fig. 11.4(a) Time domain

Example 11-1: 1 kHz Circuit

Determine load current $i_L(t)$ in the circuit of **Fig. 11-4(a)**, given that $v_s(t) = 10\cos(2\pi \times 10^3 t)$ (V), $R_1 = 5\ \Omega$, $C_1 = C_2 = 10\ \mu\text{F}$, $L_1 = 1$ mH, $L_2 = 3$ mH, $M = 0.5$ mH, and $R_L = 20\ \Omega$.

$\mathbf{V}_s = 10$



Example 11-1: 1 kHz Circuit

Determine load current $i_L(t)$ in the circuit of **Fig. 11-4(a)**, given that $v_s(t) = 10\cos(2\pi \times 10^3 t)$ (V), $R_1 = 5\ \Omega$, $C_1 = C_2 = 10\ \mu\text{F}$, $L_1 = 1$ mH, $L_2 = 3$ mH, $M = 0.5$ mH, and $R_L = 20\ \Omega$.

KVL $-10 + \left[5 + \frac{1}{j\,2\pi\times10^{-2}} + j(2\pi\times10^3)(1\times10^{-3})\right]\mathbf{I}_1 + j(2\pi\times10^3)(0.5\times10^{-3})(-\mathbf{I}_L) = 0$

$-j(2\pi\times10^3)(0.5\times10^{-3})\mathbf{I}_1 + \left(\frac{1}{j\,2\pi\times10^{-2}} + 20 + j(2\pi\times10^3)(3\times10^{-3})\right)\mathbf{I}_L = 0$

$\frac{1}{j(2\pi\times10^3)(10\times10^{-6})} = -j\frac{100}{2\pi} = -j\frac{50}{\pi}\ (= Z_C)$

$\left(\frac{1}{j} = \frac{-(-1)}{j} = \frac{-j^2}{j} = -j\right)$

$20\,\Omega$

$j\omega L_2 = j(2\pi\times10^3)(3\times10^{-3}) = j\,6\pi$

RLC $\rightarrow \left(20 - j\frac{50}{\pi} + j\,6\pi\right)\mathbf{I}_L$

$= 20 + j\left(6\pi - \frac{50}{\pi}\right)$

$$\Rightarrow -10 + \left(5 - j\frac{50}{\pi} + j2\pi\right)\mathbf{I}_1 - j\pi\,\mathbf{I}_L = 0 \qquad (a)$$

$$-j\pi\,\mathbf{I}_1 + \left(20 + j\left(6\pi - \frac{50}{\pi}\right)\right)\mathbf{I}_L = 0 \qquad (b)$$

From (b), $\mathbf{I}_1 = \dfrac{20 + j\left(6\pi - \frac{50}{\pi}\right)}{j\pi}\,\mathbf{I}_L$ (b)'

(b)' → (a) ⇒

$$-10 + \left[\left(5 + j\left(2\pi - \frac{50}{\pi}\right)\right)\left(\frac{20 + j\left(6\pi - \frac{50}{\pi}\right)}{j\pi}\right) - j\pi\right]\mathbf{I}_L = 0$$

$$\Rightarrow \mathbf{I}_L = \frac{10}{5 + j\left(2\pi - \frac{50}{\pi}\right)\left(\frac{20 + j\left(6\pi - \frac{50}{\pi}\right)}{j\pi}\right) - j\pi}$$

$(\times j\pi)$

$$= \frac{10(j\pi)}{\underbrace{5(j\pi)}_{\text{imaginary} = j5\pi} + j\left[\left(2\pi - \frac{50}{\pi}\right)\left(20 + j\left(6\pi - \frac{50}{\pi}\right)\right) - \underbrace{(j\pi)^2}_{-\pi^2}\right]}$$

imaginary $= 10\pi e^{j90°}$

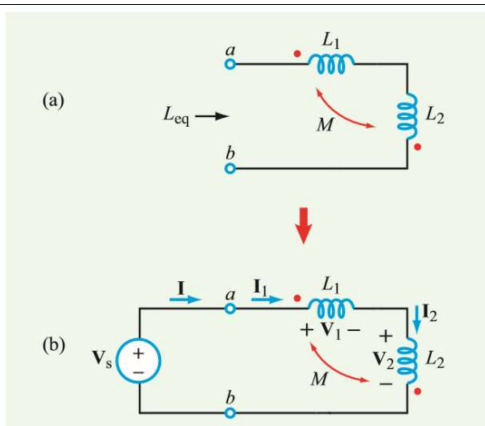
$$= \frac{10\pi e^{j90°}}{\pi^2 + j\left(5\pi + \left(2\pi - \frac{50}{\pi}\right)\left(20 + j\left(6\pi - \frac{50}{\pi}\right)\right)\right)}$$

$$= \frac{10\pi e^{j90°}}{\underbrace{\pi^2}_{\text{Real}} + j\underbrace{\left(5\pi + \left(2\pi - \frac{50}{\pi}\right)20\right)}_{I} - \left(6\pi - \frac{50}{\pi}\right)}$$

$$= \frac{\overset{10\pi}{Z_1}\, e^{j\overset{90}{\theta_1}}}{Z_2\, e^{j\theta_2}} = \left(\frac{Z_1}{Z_2}\right) e^{j(\overset{90°}{\theta_1} - \theta_2)}$$

$Z_2 = \sqrt{R^2 + I^2}$ $\quad \theta_2 = \tan^{-1}\frac{I}{R}$

$$i_L(t) = \left(\frac{Z_1}{Z_2}\right)\cos\left(2\pi \times 10^3 t + (90° - \theta_2°)\right)$$



$$\mathbf{I}_1 = \mathbf{I}_2 = \mathbf{I},$$

and while $\mathbf{I}_1$ enters L_1 at its dotted terminal, $\mathbf{I}_2$ enters L_2 at its undotted terminal. While guided by **Fig. 11-3(b)**, application of the dot convention to the loop in **Fig. 11-6(b)** gives

$$\mathbf{V}_1 = j\omega L_1\mathbf{I}_1 - j\omega M\mathbf{I}_2 = j\omega(L_1 - M)\mathbf{I}$$

and (note the • convention)

$$\mathbf{V}_2 = j\omega L_2\mathbf{I}_2 - j\omega M\mathbf{I}_1. = j\omega(L_2 - M)\mathbf{I}.$$

At terminals (a, b),

$$\mathbf{V}_s = \mathbf{V}_1 + \mathbf{V}_2 = j\omega(L_1 + L_2 - 2M)\mathbf{I}.$$

For the circuit in circuit **Fig. 11-6(c)**,

$$\mathbf{V}_s = j\omega L_{eq}\mathbf{I}.$$

Equivalency leads to

$$L_{eq} = L_1 + L_2 - 2M.$$

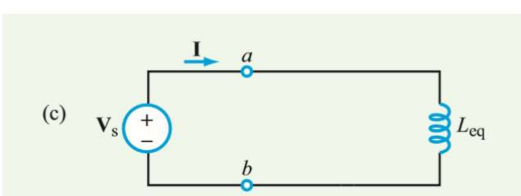


Figure 11-6: Finding L_{eq} of two series-coupled inductors (Example 11-3).

$L_{eq} = L_1 + L_2 - 2M$

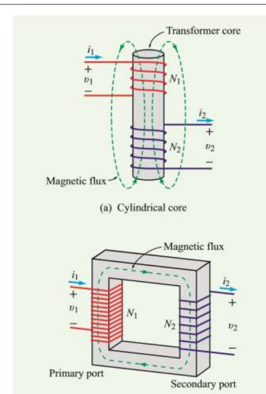
11-2 Transformers

11-2.1 Coupling Coefficient

To couple magnetic flux between two coils, the coils may be wound around a common core (**Fig. 11-7(a)**), on two separate arms of a rectangular core (**Fig. 11-7(b)**), or in any other arrangement conducive to having a significant fraction of the magnetic flux generated by each coil shared with the other. The ***coupling coefficient*** k defines the degree of magnetic coupling between the coils, with $0 \le k \le 1$. For a ***loosely coupled*** pair of coils, $k < 0.5$; for ***tightly coupled*** coils, $k > 0.5$; and for ***perfectly coupled*** coils, $k = 1$. The magnitude of k depends on the physical geometry of the two-coil configuration and the magnetic permeability μ of the core material.

► A transformer is said to be ***linear*** if μ of its core material is a constant, independent of the magnitude of the currents flowing through the coils (and hence, the strength of the induced magnetic field). ◄

Most core materials, including air, wood, and ceramics, are nonferromagnetic, and their μ is approximately equal to μ_0, the ***permeability of free space***. When nonferromagnetic materials are used for the common core around which the coils are wound, the magnitude of k depends entirely on how tightly coupled the two windings are. Such transformers are indeed linear, but the magnitude of k is seldom greater than 0.4. Increasing k requires the use of ferromagnetic cores, but the transformer becomes heavier in weight and its behavior becomes nonlinear.



(a) Cylindrical core

(b) Rectangular core

As was noted earlier in connection with **Fig. 11-1(a)**, current i_1, through coil 1 generates magnetic fluxes Φ_{11} through coil 1 and Φ_{12} through both coils 1 and 2. The coupling coefficient is given by

$$k = \frac{\Phi_{12}}{\Phi_{11} + \Phi_{12}} = \frac{\Phi_{12}}{\Phi_1} . \qquad (11.20a)$$

where $\Phi_1 = \Phi_{11} + \Phi_{12}$. The perfectly coupled case corresponds to when the flux coupled to coil 2, namely Φ_{12}, is equal to the self-coupled flux Φ_1. Similarly, from the standpoint of coil 2,

$$k = \frac{\Phi_{21}}{\Phi_{22} + \Phi_{21}} = \frac{\Phi_{21}}{\Phi_2} . \qquad (11.20b)$$

Through energy considerations, k can be related to L_1, L_2, and M as

$$k = \frac{M}{\sqrt{L_1 L_2}} . \qquad (11.21)$$

The mutual inductance M is a maximum when $k = 1$ (perfectly coupled transformer),

$$M(\max) = \sqrt{L_1 L_2} . \qquad (11.22)$$

(perfectly coupled transformer with $k = 1$)

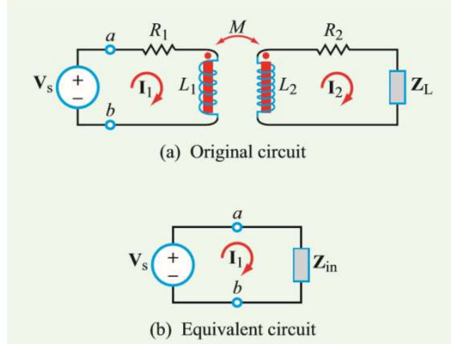


Figure 11-8: (a) Transformer circuit with coil resistors R_1 and R_2, and (b) in terms of an equivalent input impedance $\mathbf{Z}_{in}$.

$$-\mathbf{V}_s + (R_1 + j\omega L_1)\mathbf{I}_1 - j\omega M\mathbf{I}_2 = 0 \qquad (11.23a)$$

and

$$-j\omega M\mathbf{I}_1 + (R_2 + j\omega L_2 + \mathbf{Z}_L)\mathbf{I}_2 = 0. \qquad (11.23b)$$

From the standpoint of source $\mathbf{V}_s$, the circuit to the right of terminals (a, b) can be represented by an equivalent ***input impedance*** $\mathbf{Z}_{in}$, as depicted in **Fig. 11-8(b)**. By manipulating Eqs. (11.23) to eliminate $\mathbf{I}_2$, we can generate the following expression for $\mathbf{Z}_{in}$:

$$\mathbf{Z}_{in} = \frac{\mathbf{V}_s}{\mathbf{I}_1} = (R_1 + j\omega L_1) + \frac{\omega^2 M^2}{R_2 + j\omega L_2 + \mathbf{Z}_L}$$
$$= (R_1 + j\omega L_1) + \mathbf{Z}_R, \qquad (11.24)$$

where we define the second term as the ***reflected impedance*** $\mathbf{Z}_R$, namely

$$\mathbf{Z}_R = \frac{\omega^2 M^2}{R_2 + j\omega L_2 + \mathbf{Z}_L} . \qquad (11.25)$$

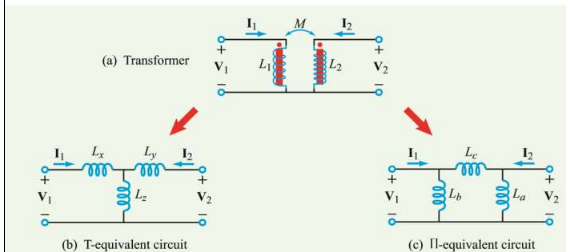


Figure 11-10: The transformer can be modeled in terms of T- or Π-equivalent circuits.

Transformer dots on same ends

$$L_x = L_1 - M, \qquad (11.29a)$$

$$L_y = L_2 - M, \qquad (11.29b)$$

and

$$L_z = M. \qquad (11.29c)$$

Transformer with dots on same ends

$$L_a = \frac{L_1 L_2 - M^2}{L_1 - M} , \qquad (11.31a)$$

$$L_b = \frac{L_1 L_2 - M^2}{L_2 - M} , \qquad (11.31b)$$

and

$$L_c = \frac{L_1 L_2 - M^2}{M} . \qquad (11.31c)$$

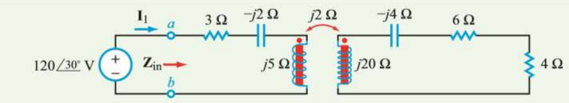


Figure 11-9: Circuit of Example 11-4.

Example 11-5: Equivalent Circuit

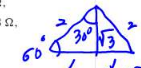
Use the T-equivalent circuit model to determine $\mathbf{I}_1$ in the circuit of **Fig. 11-11**.

Solution: Use of Eq. (11.29) gives

$$j\omega L_x = j\omega L_1 - j\omega M = j5 - j2 = j3\ \Omega,$$
$$j\omega L_y = j\omega L_2 - j\omega M = j20 - j2 = j18\ \Omega,$$

and

$$j\omega L_z = j\omega M = j2\ \Omega.$$



$\cos 30° = \frac{\sqrt{3}}{2}$
$\sin 30° = \frac{1}{2}$

The T-equivalent circuit is shown in **Fig. 11-11(b)**. Application of the mesh analysis by-inspection method leads to the matrix equation

$$\begin{bmatrix} (3 - j2 + j3 + j2) & -j2 \\ -j2 & (j2 + j18 - j4 + 6 + 4) \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 120e^{j30°} \\ 0 \end{bmatrix}.$$

$60\sqrt{3} + j60$

$$\mathbf{I}_1 = \frac{\begin{vmatrix} 120e^{j30°} & -2 \\ 0 & 10+j16 \end{vmatrix}}{\begin{vmatrix} 3+j3 & -j2 \\ -j2 & 10+j16 \end{vmatrix}}$$

Its solution is

$$\mathbf{I}_1 = 28.6e^{-j12.2°}\ \text{A},$$

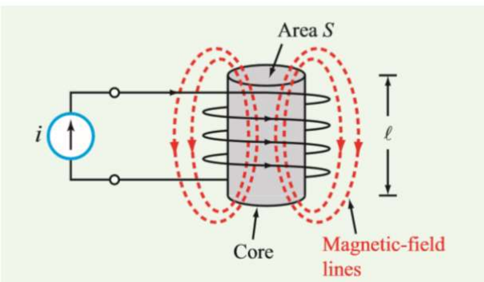


Figure 5-20: The inductance of a solenoid of length ℓ and cross-sectional area S is $L = \mu N^2 S/\ell$, where N is the number of turns and μ is the magnetic permeability of the core material.

11-4 Ideal Transformers

An ideal transformer is characterized by lossless coils ($R_1 = R_2 = 0$) and a maximum coupling coefficient $k = 1$. The mutual inductance M is a maximum and given by Eq. (11.22) as

$$M(\max) = \sqrt{L_1 L_2}\,. \qquad \textbf{(ideal transformer)} \qquad (11.34)$$

According to Eq. (5.52), the inductance L of a solenoid-shaped inductor is proportional to N^2, where N is the number of turns wound around the core. Hence, for an ideal transformer with N_1 turns on the primary side and N_2 on the secondary, as depicted in **Fig. 11-14**,

$$\frac{L_2}{L_1} = \frac{N_2^2}{N_1^2} = n^2, \qquad (11.35)$$

where $n = N_2/N_1$ is called the ***turns ratio***.

► The transformer is called a ***step-up transformer*** ($V_2 > V_1$) when $n > 1$ and a ***step-down transformer*** ($V_2 < V_1$) when $n < 1$. ◄

For a lossless transformer, complex power $\mathbf{S}_1$ supplied by its primary side must match complex power $\mathbf{S}_2$ absorbed by its secondary:

$$\mathbf{S}_1 = \mathbf{S}_2, \qquad (11.37)$$

or equivalently

$$\mathbf{V}_1\mathbf{I}_1^* = \mathbf{V}_2\mathbf{I}_2^*. \qquad (11.38)$$

In view of Eq. (11.36), it follows that

$$\frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{1}{n} \qquad \textbf{(ideal transformer with dots on same ends).} \qquad (11.39)$$

The expressions for the voltage and current ratios given by Eqs. (11.36) and (11.39) apply to the current directions, voltage polarities and dot locations indicated in both configurations of **Fig. 11-14**.

Self-inductance refers to the magnetic-flux linkage of a coil (or circuit) with itself, in contrast with ***mutual inductance***, which refers to magnetic-flux linkage in a coil due to the magnetic field generated by another coil (or circuit). Usually, when the term ***inductance*** is used, the intended reference is to self-inductance. Mutual inductance is covered in Chapter 11.

The (self) inductance of any conducting system is defined as the ratio of Λ to the current i responsible for generating it, given as

$$L = \frac{\Lambda}{i} \quad \text{(H)}, \qquad (5.51)$$

and its unit is the henry (H), so named to honor the American inventor Joseph Henry (1797–1878). Using the expression for Λ given by Eq. (5.50), we have

$$L = \frac{\mu N^2 S}{\ell} \qquad \textbf{(solenoid).} \qquad (5.52)$$

The inductance L is directly proportional to μ, the magnetic permeability of the core material. The relative magnetic permeability μ_r is defined as

$$\mu_r = \frac{\mu}{\mu_0}, \qquad (5.53)$$

where $\mu_0 \approx 4\pi \times 10^{-7}$ (H/m) is the magnetic permeability of free space.