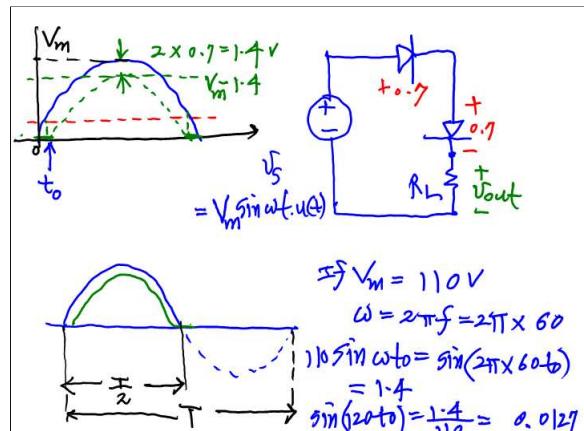
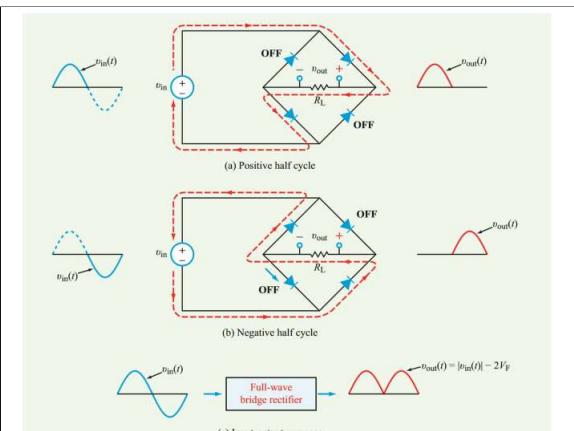
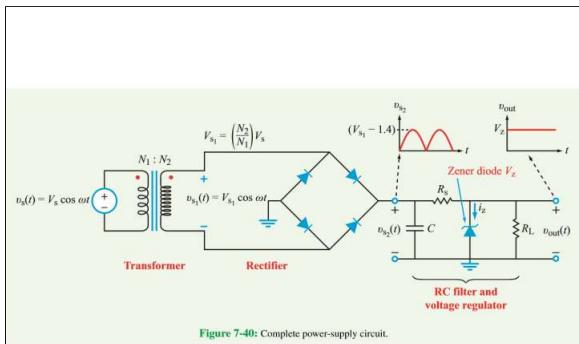
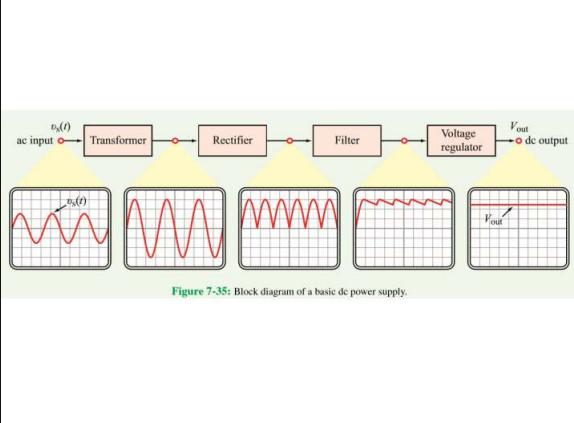
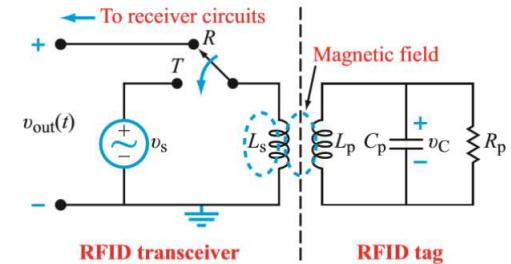


ECE 101 F19 Lecture 15/F. Nov. 19, 2019

HW #8 for Quiz 8 on Nov. 21

Quiz 9 will be given on Dec 3
and covers both HW9 & HW10

- | | |
|-------------|-------------------------|
| 1) Prob 6.1 | 8) 6.18 |
| 2) 6.3 | 9) 6.22 |
| 3) 6.7 | 10) 6.28 |
| 4) 6.10 | |
| 5) 6.12 | Midterm avg 57.5 |
| 6) 6.14 | 0 ~ 17.6 |
| 7) 6.16 | Highest 92
Lowest 20 |



$$\Rightarrow 2\pi \times 60 \times t_0 = 0.0127 \text{ Rad}$$

$$t_0 = \frac{0.0127 \text{ Rad}}{2\pi \times 60} = 0.000037 \text{ sec}$$

$$T = \frac{1}{f} = \frac{1}{60}$$

$$\frac{T}{2} = \frac{1}{2} \left(\frac{1}{60} \right) = 8.3 \text{ mS} = 833 \mu\text{s}$$

$$\frac{34}{833} = 0.041 \text{ (} \pm 1\% \text{)}$$

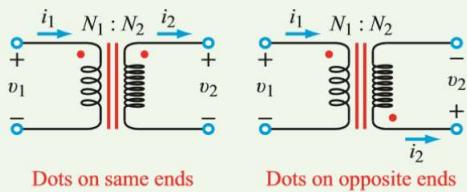
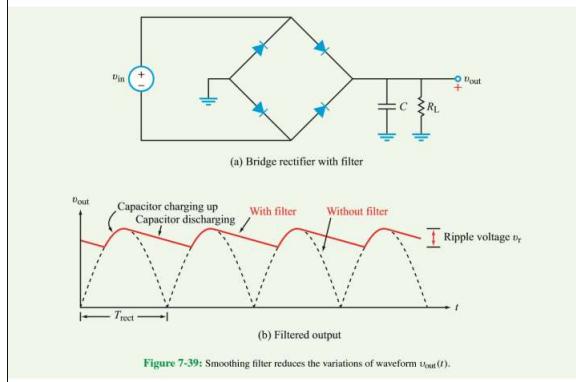


Figure 7-36: Schematic symbol for an ideal transformer. Note the reversal of the voltage polarity and current direction when the dot location at the secondary is moved from the top end of the coil to the bottom end. For both configurations:

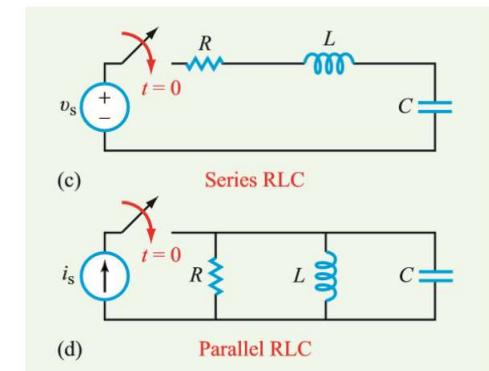
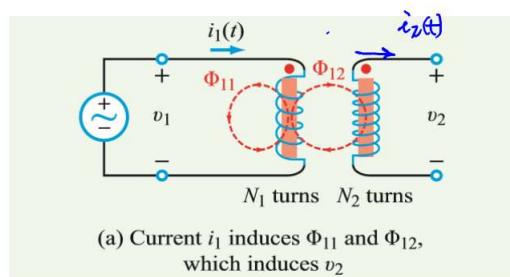
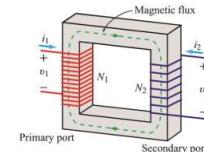
$$\frac{v_2}{v_1} = \frac{N_2}{N_1} = n, \quad \frac{i_2}{i_1} = \frac{N_1}{N_2} = \frac{1}{n}, \quad \frac{p_2}{p_1} = \frac{v_2 i_2}{v_1 i_1} = 1$$

Magnetically Coupled Circuits

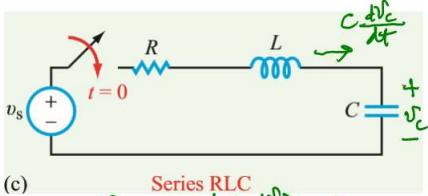
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Objectives



Solving for $v_C(t)$:



(c) Series RLC

$$\begin{aligned} \dot{v}_s &= R C \frac{dv_C}{dt} + L \frac{di_L}{dt} (C \frac{dv_C}{dt}) + v_C \\ \div LC &\Rightarrow \frac{d^2v_C}{dt^2} + \frac{1}{L} \frac{dv_C}{dt} + \frac{1}{C} v_C = \frac{\dot{v}_s}{LC} \end{aligned}$$

If $R=0$, $\alpha=0$ (no damping)

$$\text{damping coefficient } \alpha = \frac{R}{2L} \quad (\text{Np/s}), \quad (6.1\text{a})$$

$$\text{resonant frequency } \omega_0 = \frac{1}{\sqrt{LC}} \quad (\text{rad/s}). \quad (6.1\text{b})$$

(series RLC)

Overdamped response

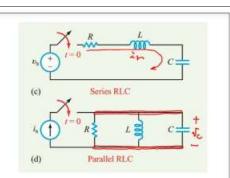
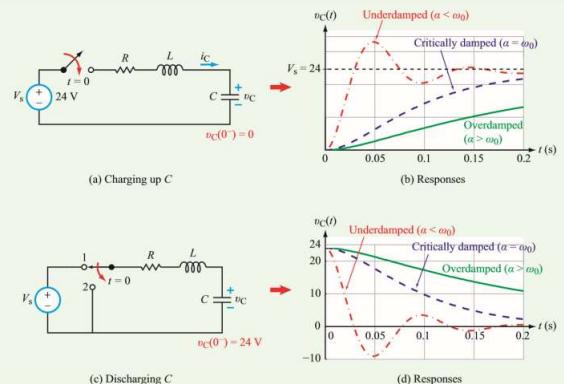
Critically damped response

Underdamped response

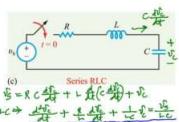
$$\alpha > \omega_0,$$

$$\alpha = \omega_0,$$

$$\alpha < \omega_0.$$

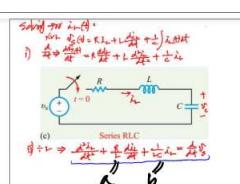


Solving for $v_C(t)$:

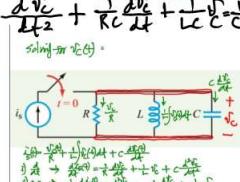


(c) Series RLC

$$\div LC \Rightarrow \frac{d^2v_C}{dt^2} + \frac{1}{R} \frac{dv_C}{dt} + \frac{1}{LC} v_C = \frac{1}{C} \frac{di_R}{dt}$$

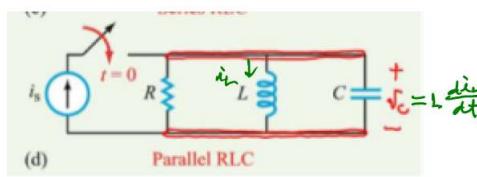


$$\frac{d^2v_C}{dt^2} + \frac{1}{RC} \frac{dv_C}{dt} + \frac{1}{LC} v_C = \frac{1}{C} \frac{di_R}{dt}$$



$$\frac{d^2v_C}{dt^2} + \frac{1}{RC} \frac{dv_C}{dt} + \frac{1}{LC} v_C = -\frac{1}{C} \frac{di_R}{dt}$$

$$\text{KCL: } \frac{1}{Z} = \frac{L \frac{di_L}{dt}}{R} + i_R + C \frac{1}{dt} \left(L \frac{di_L}{dt} \right) - \frac{1}{R} \frac{di_R}{dt} + i_L + LC \frac{1}{dt} \frac{dv_C}{dt}$$



$$\frac{1}{Z} \frac{di_L}{dt} + \frac{1}{RC} \frac{dv_C}{dt} + \frac{1}{LC} i_R = -\frac{1}{C} \frac{di_R}{dt}$$

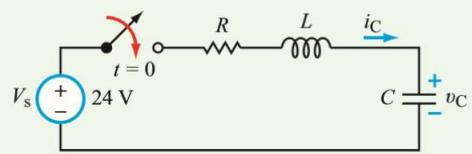


Figure 6-7: Series RLC circuit connected to a source V_s at $t = 0$. In general, the capacitor may have had an initial charge on it at $t = 0^-$, with a corresponding initial voltage $v_C(0^-)$.

$$\frac{d^2v_C}{dt^2} + \frac{R}{L} \frac{dv_C}{dt} + \frac{1}{LC} v_C = \frac{V_s}{LC}. \quad (6.4)$$

For convenience, we rewrite Eq. (6.4) in the abbreviated form

$$v''_C + av'_C + bv_C = c, \quad (6.5)$$

where

$$a = \frac{R}{L}, \quad b = \frac{1}{LC}, \quad c = \frac{V_s}{LC}. \quad (6.6)$$

$$v_t(t) = Ae^{st}, \quad (6.11)$$

where A and s are constants to be determined later. To ascertain that Eq. (6.11) is indeed a viable solution of Eq. (6.8), we insert the proposed expression for $v_t(t)$ and its first and second derivatives in Eq. (6.8). The result is

$$s^2 Ae^{st} + asAe^{st} + bAe^{st} = 0, \quad (6.12)$$

which simplifies to

$$s^2 + as + b = 0. \quad (6.13)$$

Hence, the proposed solution given by Eq. (6.11) is indeed an acceptable solution so long as Eq. (6.13) is satisfied.

The quadratic equation given by Eq. (6.13) is known as the **characteristic equation** of the differential equation. It has two roots:

$$s_1 = -\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b}, \quad (6.14a)$$

$$s_2 = -\frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b}. \quad (6.14b)$$

6-3.1 Differential Equation

For the circuit in **Fig. 6-7**, the KVL loop equation for $t \geq 0$ (after closing the switch) is

$$Ri_C + L \frac{di_C}{dt} + v_C = V_s \quad (\text{for } t \geq 0), \quad (6.2)$$

6-3.2 Solution of Differential Equation

The general solution of the second-order differential equation given by Eq. (6.5) consists of two components:

$$v_C(t) = v_{tr}(t) + v_{ss}(t), \quad (6.7)$$

where $v_{tr}(t)$ is the **transient** (also called **homogeneous**) solution of Eq. (6.5) or the **natural response** of the RLC circuit and $v_{ss}(t)$ is the **steady-state** solution (also called **particular** solution). The transient solution is the solution of Eq. (6.5) under source-free conditions; i.e., with $V_s = 0$, which means that $c = V_s/LC$ also is zero. Thus $v_{tr}(t)$ is the solution of

$$v''_{tr} + av'_{tr} + bv_{tr} = 0 \quad (\text{source-free}). \quad (6.8)$$

$$\begin{aligned} s^2 + as + b &= 0 \Leftrightarrow x'' + ax' + b = 0 \\ s &= \frac{-a \pm \sqrt{a^2 - 4b}}{2} = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b} \\ s_1 &= -\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b}, \quad s_2 = -\frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b} \end{aligned}$$

	x	R	b	$a = \frac{R}{L}$	$\omega = \sqrt{b}$
<u>Series RLC</u>	v_C	$\frac{R}{L}$	$\frac{1}{LC}$	$\frac{R}{2L}$	$\frac{1}{\sqrt{LC}}$
	i_L	$\frac{1}{R}$	$\frac{1}{LC}$	$\frac{1}{2R}$	$\frac{1}{\sqrt{LC}}$
<u>Parallel RLC</u>	v_C	$\frac{1}{RC}$	$\frac{1}{LC}$	$\frac{1}{2RC}$	$\frac{1}{\sqrt{LC}}$

$$s_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2}$$

① $\lambda > \omega_0$ $s_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2}$ s_1, s_2 are real numbers
overdamped

② $\lambda = \omega_0$ $s_{1,2} = -\lambda$ $s_1 = s_2 = -\lambda$
critically damped

③ $\lambda < \omega_0$ $s_{1,2} = -\lambda \pm j\sqrt{\omega_0^2 - \lambda^2}$
underdamped s_1, s_2 are complex numbers

The existence of two distinct roots implies that Eq. (6.8) has two viable solutions, one in terms of $e^{s_1 t}$ and another in terms of $e^{s_2 t}$. Hence, we should generalize the form of our solution to

$$v_{tr}(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{for } t \geq 0, \quad (6.15)$$

where constants A_1 and A_2 are to be determined shortly.

Inserting Eq. (6.15) into Eq. (6.10) leads to

$$v_C(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + v_C(\infty). \quad (6.16)$$

$$v_C(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + v_C(\infty) \quad (1)$$

At $t = 0$ $v_C(0) = A_1 + A_2 + v_C(\infty) \quad (= v_C(0)) \quad (2)$

$\frac{dv_C(t)}{dt} \Big|_{t=0} = A_1 s_1 + A_2 s_2 + 0 = \frac{1}{C} i_C(t) \quad (3)$

(2) $\times s_1 \Rightarrow s_1 v_C(0) = A_1 s_1 + A_2 s_1 + s_1 v_C(\infty) \quad (4)$

(3) $\underline{-s_1 v_C(0)} = A_1 s_1 + A_2 s_2$

$v_C'(0) - s_1 v_C(0) = A_2 (s_2 - s_1) - s_1 v_C(\infty)$

$\Rightarrow A_2 = \frac{s_1(v_C(0) - v_C(\infty)) + v_C'(0)}{s_2 - s_1} = \frac{1}{C} i_C(0) \quad (5)$

From (2) and (5), $A_1 = v_C(0) - A_2 - v_C'(0)$

$$\Rightarrow A_2 = \frac{s_1(v_C(0) - v_C(\infty)) + v_C'(0)}{s_2 - s_1} = \frac{1}{C} i_C(0) \quad (5)$$

From (2) and (5), $A_1 = v_C(0) - A_2 - v_C'(0)$

$$= v_C(0) + \frac{s_1(v_C(0) - v_C(\infty)) - \frac{1}{C} i_C(0)}{s_2 - s_1} - v_C'(0)$$

$$= \frac{v_C(0)(s_2 - s_1) + s_1(v_C(0) - v_C(\infty)) - \frac{1}{C} i_C(0) - \frac{1}{C} i_C(0)(s_2 - s_1)}{s_2 - s_1}$$

$$= \frac{-\frac{1}{C} i_C(0) + s_2(v_C(0) - v_C(\infty))}{s_2 - s_1} = A_1$$

6-3 Invoking Initial Conditions

To determine the values of constants A_1 and A_2 in Eq. (6.16), we need to **invoke initial conditions**, which means that we need to use information available to us about the values of v_C and its time derivative v_C' , both at $t = 0$. Since

$$i_C(t) = C \frac{dv_C}{dt} = C v'(t), \quad (6.19)$$

the second requirement is equivalent to needing to know $i_C(0)$.

At $t = 0$, Eq. (6.16) simplifies to

$$v_C(0) = A_1 + A_2 + v_C(\infty), \quad (6.20)$$

and

$$i_C(0) = C \frac{dv_C}{dt} \Big|_{t=0} = C(s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}) \Big|_{t=0}$$

$$= C(s_1 A_1 + s_2 A_2). \quad (6.21)$$

Simultaneous solution of Eqs. (6.20) and (6.21) for A_1 and A_2 gives

$$A_1 = \frac{\frac{1}{C} i_C(0) - s_2[v_C(0) - v_C(\infty)]}{s_1 - s_2}, \quad (6.22a)$$

$$A_2 = \frac{\frac{1}{C} i_C(0) - s_1[v_C(0) - v_C(\infty)]}{s_2 - s_1}. \quad (6.22b)$$

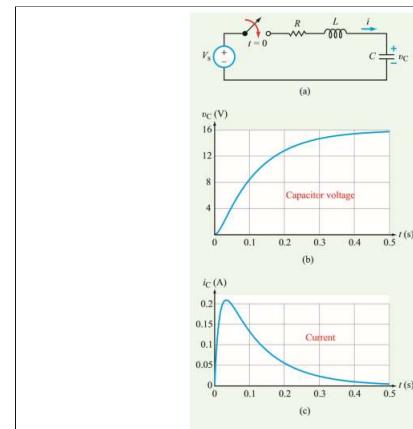


Figure 6-8: Example 6-3: (a) circuit, (b) $v_C(t)$, and (c) $i_C(t)$.

Table 6-1: Step response of RLC circuits for $t > 0$.	
Series RLC	Parallel RLC
Total Response	Total Response
Overshooted ($\omega > \alpha$)	Overshooted ($\omega > \alpha$)
$v_C(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + v_C(\infty)$	$i_L(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + i_L(\infty)$
$A_1 = \frac{1}{\lambda_1 - \omega} [v_C(0) - v_C(\infty)]$	$A_1 = \frac{1}{\lambda_1 - \omega} [v_L(0) - v_L(\infty)]$
$A_2 = \left[\frac{1}{\lambda_2 - \omega} [v_C(0) - v_C(\infty)] - \frac{A_1}{\lambda_2 - \omega} \right]$	$A_2 = \left[\frac{1}{\lambda_2 - \omega} [v_L(0) - v_L(\infty)] - \frac{A_1}{\lambda_2 - \omega} \right]$
Critically Damped ($\omega = \alpha$)	Critically Damped ($\omega = \alpha$)
$v_C(t) = (B_1 + B_2 t)e^{-\alpha t} + v_C(\infty)$	$i_L(t) = (B_1 + B_2 t)e^{-\alpha t} + i_L(\infty)$
$B_1 = v_C(0) - v_C(\infty)$	$B_1 = i_L(0) - i_L(\infty)$
$B_2 = \frac{1}{\alpha} [v_C(0) + \alpha(v_C(0) - v_C(\infty))]$	$B_2 = \frac{1}{\alpha} [v_L(0) + \alpha(i_L(0) - i_L(\infty))]$
Underdamped ($\omega < \alpha$)	Underdamped ($\omega < \alpha$)
$v_C(t) = e^{-\alpha t} (D_1 \cos \omega t + D_2 \sin \omega t) + v_C(\infty)$	$i_L(t) = e^{-\alpha t} (D_1 \cos \omega t + D_2 \sin \omega t) + i_L(\infty)$
$D_1 = v_C(0) - v_C(\infty)$	$D_1 = i_L(0) - i_L(\infty)$
$D_2 = \frac{1}{\omega} [v_C(0) + \alpha(v_C(0) - v_C(\infty))]$	$D_2 = \frac{1}{\omega} [v_L(0) + \alpha(i_L(0) - i_L(\infty))]$
Auxiliary Relations	
$\sigma = \begin{cases} \frac{\omega}{2L} & \text{Series RLC} \\ \frac{1}{2\omega C} & \text{Parallel RLC} \end{cases}$	$\omega_D = \sqrt{\omega_0^2 - \alpha^2}$
$\lambda_1 = -\alpha - \sqrt{\alpha^2 - \omega_D^2}$	$\omega_D = \sqrt{\omega_0^2 - \alpha^2}$
$\lambda_2 = -\alpha + \sqrt{\alpha^2 - \omega_D^2}$	$\lambda_2 = -\alpha - \sqrt{\alpha^2 - \omega_D^2}$

critically damped case

$$s^2 + \alpha s + b = (s - \lambda_1)(s - \lambda_2) \Rightarrow \lambda_1 = \lambda_2 = -\alpha$$

$$\Rightarrow \frac{A_1}{(s - \lambda_1)} + \frac{A_2}{(s - \lambda_2)} \leftrightarrow \frac{B_1}{(s + \alpha)} + \frac{B_2}{(s + \alpha)}$$

$$\downarrow L^+$$

$$A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} \leftrightarrow B_1 e^{-\alpha t} + B_2 t e^{-\alpha t}$$

$$v_C(t) = B_1 e^{-\alpha t} + B_2 t e^{-\alpha t} + v_C(\infty)$$

$$v_C(\infty) = B_1 + B_2 \alpha$$

$$\frac{1}{s} v_C(s)|_{t=0} = -B_1 - B_2 \alpha + v_C(\infty)$$

$$= -\alpha B_1 + B_2 = v_C'(0) = \frac{1}{s} v_C(s)|_{t=0}$$

$$v_C(s) = B_1 + v_C(\infty)$$

$$\frac{1}{s} v_C(s)|_{t=0} = -B_1 e^{-\alpha t} + B_2 e^{-\alpha t} + B_2 t e^{-\alpha t}|_{t=0}$$

$$= -\alpha B_1 + B_2 = v_C'(0) = \frac{1}{s} v_C(s)|_{t=0}$$

From (1), $B_1 = v_C(0) - v_C(\infty)$

From (2), $B_2 = \alpha B_1 + \frac{1}{s} v_C(s)|_{t=0} = \underline{(v_C(0) - v_C(\infty))} + \underline{\frac{1}{s} v_C(s)|_{t=0}}$

underdamped case ($\omega < \omega_0$)

$$\lambda_1, \lambda_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_D^2}$$

$$= -\alpha \pm j \sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j \omega_D$$

$$\omega_D = \sqrt{\omega_0^2 - \alpha^2}$$

damped frequency

$$v_C(t) = C_1 e^{-(\alpha+j\omega_D)t} + C_2 e^{-(\alpha-j\omega_D)t} + v_C(\infty) \quad (*)$$

$$v_C(0) = C_1 + C_2 + v_C(\infty) \quad (1)$$

$$\frac{1}{s} v_C(s)|_{t=0} = -(\alpha + j\omega_D) C_1 - (\alpha - j\omega_D) C_2 \quad (2)$$

$$(1) \times (\alpha + j\omega_D) \rightarrow (\alpha + j\omega_D) v_C(0) = (\alpha + j\omega_D) C_1 + (\alpha + j\omega_D) C_2$$

$$(2) \rightarrow \frac{1}{s} v_C(s)|_{t=0} = - (\alpha + j\omega_D) C_1 + (\alpha - j\omega_D) C_2$$

$$(\alpha + j\omega_D) v_C(0) + \frac{1}{s} v_C(s)|_{t=0} = -j \omega_D C_1 + (\alpha + j\omega_D) v_C(0)$$

$$\Rightarrow C_2 = \frac{(\alpha + j\omega_D) v_C(0) + \frac{1}{s} v_C(s)|_{t=0}}{-j \omega_D} = \frac{-j \omega_D C_1 + (\alpha + j\omega_D) v_C(0) + \frac{1}{s} v_C(s)|_{t=0}}{j \omega_D}$$

$$= \frac{-\alpha (v_C(0) - v_C(\infty)) + j\omega_D (v_C(0) - v_C(\infty)) + \frac{1}{s} v_C(s)|_{t=0}}{j \omega_D} = C_2 \quad (3)$$

From (1), $C_1 + C_2 = v_C(0) - v_C(\infty)$

$$C_1 = v_C(0) - v_C(\infty) - C_2, \text{ and}$$

$$\frac{1}{s} (v_C(0) - v_C(\infty)) - j \frac{1}{s} \frac{\alpha (v_C(0) - v_C(\infty)) + \frac{1}{s} v_C(s)|_{t=0}}{\omega_D} = C_2 \quad (3)$$

thus $C_1 = \frac{1}{s} (v_C(0) - v_C(\infty)) + j \frac{1}{s} \frac{\alpha (v_C(0) - v_C(\infty)) + \frac{1}{s} v_C(s)|_{t=0}}{\omega_D} \quad (4)$

$$= \alpha + j b = \sqrt{a^2 + b^2} e^{j \tan^{-1} \frac{b}{a}} = |C_1| e^{j \theta}$$

b a $C_2 = C_1^* = |C_1| e^{-j \theta}$

Finally,

$$\begin{aligned} v_c(t) &= C_1 e^{-(\omega+j\omega)t} + C_2 e^{-(\omega-j\omega)t} \\ &= |C_1| e^{[(-\omega+j\omega)t-\theta]} + |C_2| e^{[(-\omega-j\omega)t+\theta]} \\ &= |C_1| e^{-\omega t} \left(e^{-j(\omega t-\theta)} + e^{+j(\omega t-\theta)} \right) \\ &= |C_1| e^{-\omega t} \times \cos(\omega t - \theta) \end{aligned}$$

From (2), (5), (6) \rightarrow

$$\begin{aligned} v_c(t) &= k e^{-\omega t} \left(\cos(\omega t - \theta) + \sin(\omega t - \theta) \right) \\ &= 2 \cos(\omega t - \theta) \\ &= 2 \cos \omega t + \cos \theta + \sin \omega t + \sin \theta \end{aligned}$$

$$a = \frac{1}{2} (\Re(v_c(0)) - \Im(v_c(0))) + j \frac{1}{2} \Im(v_c(0))$$

$$b = \frac{1}{\omega} \frac{\alpha (v_c(0) - v_c(0)) + \frac{1}{2} i v_c'(0)}{2}$$

$$\text{thus } v_c(t) - v_c(0) = |C_1| e^{-\omega t} \left[2 \left(\cos \omega t + \cos \theta + \sin \omega t + \sin \theta \right) \right]$$

$$\begin{aligned} |C_1| \cos \theta &= a \\ |C_1| \sin \theta &= b \end{aligned}$$

$$= e^{-\omega t} \left(D_1 \cos \omega t + D_2 \sin \omega t \right)$$

$e^{-at} u(t)$	\leftrightarrow	$\frac{1}{s+a}$
$e^{-a(t-T)} u(t-T)$	\leftrightarrow	$\frac{e^{-Ts}}{s+a}$
$t u(t)$	\leftrightarrow	$\frac{1}{s^2}$
$(t-T) u(t-T)$	\leftrightarrow	$\frac{e^{-Ts}}{s^2}$
$t^2 u(t)$	\leftrightarrow	$\frac{2}{s^3}$
$t e^{-at} u(t)$	\leftrightarrow	$\frac{1}{(s+a)^2}$
$t^2 e^{-at} u(t)$	\leftrightarrow	$\frac{2}{(s+a)^3}$
$t^{n-1} e^{-at} u(t)$	\leftrightarrow	$\frac{(n-1)!}{(s+a)^n}$
$\sin \omega t u(t)$	\leftrightarrow	$\frac{\omega}{s^2 + \omega^2}$
$\sin(\omega t + \theta) u(t)$	\leftrightarrow	$\frac{s \sin \theta + \omega \cos \theta}{s^2 + \omega^2}$
$\cos \omega t u(t)$	\leftrightarrow	$\frac{s \cos \theta - \omega \sin \theta}{s^2 + \omega^2}$
$\cos(\omega t + \theta) u(t)$	\leftrightarrow	$\frac{s^2 + \omega^2}{s^2 + \omega^2}$
$e^{-at} \sin \omega t u(t)$	\leftrightarrow	$\frac{\omega}{(s+a)^2 + \omega^2}$
$e^{-at} \cos \omega t u(t)$	\leftrightarrow	$\frac{s}{(s+a)^2 + \omega^2}$

$$i(0) = 0, \quad \mathcal{L}[i(t)] = L \frac{di}{dt}|_{t=0}$$

For $t > 0$,

$$\begin{aligned} \text{At } t=0, \quad 0 &= R_i(0) + V_L(0) + L \frac{di}{dt}|_{t=0} \\ \Rightarrow L \frac{di}{dt}|_{t=0} &= 1.6 = R_i(0) + V_L(0) + L \frac{di}{dt}|_{t=0} \\ i'(0) &= 1.6 / 0.4 = 4 \end{aligned}$$

$$v_s(t) = V_0 u(t)$$

$$\begin{aligned} \text{For } t > 0, \quad 1.6 &= R_i + \frac{1}{C} \int i dt + L \frac{di}{dt} \\ \frac{di}{dt} &\Rightarrow 0 = R \frac{di}{dt} + \frac{1}{C} i + L \frac{di}{dt} \\ \boxed{\frac{R}{L} \frac{di}{dt} + \frac{1}{L} \frac{di}{dt} + \frac{1}{C} i = 0} \end{aligned}$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{1}{C} i = 0$$

$$\frac{d^2 i}{dt^2} + 4 \frac{di}{dt} + 10 i = 0$$

$$s^2 + (4s + 10) = 0$$

$$s^2 + 4s + 25 = 0$$

$$\text{where } i(0) = 0, \quad i'(0) = 4$$

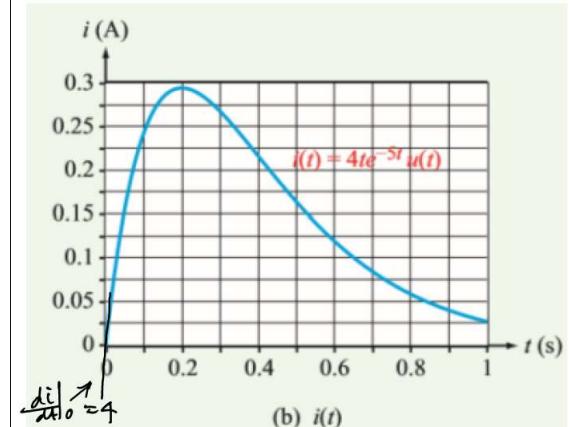
$$\frac{I(s)}{s} = \frac{4}{s^2 + 4s + 25} = \frac{4}{(s+2)^2 + 21}$$

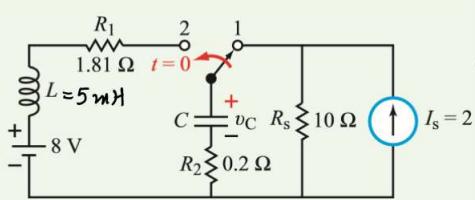
$$\left(\frac{R}{L} = \frac{4}{0.4} = 10 \right) \quad \left(\frac{1}{C} = \frac{1}{0.4 \times 0.1} = 25 \right)$$

$$i(t) = 4te^{-5t} u(t)$$

$$\frac{di}{dt}|_{t=0} = 4$$

$$\frac{di}{dt} = 4e^{-5t} + 4t(-5)e^{-5t}|_{t=0} = 4 + 0 = 4$$

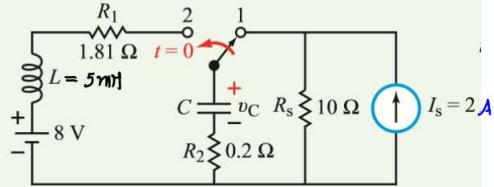




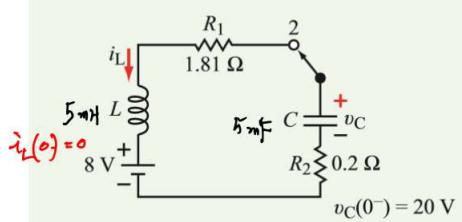
(a) Original circuit

$$i_L(0) \approx 0$$

what is $v_C(0^-)$ = ?



(a) Original circuit



$$\begin{aligned}
 & i_L(0) = 0 \\
 & v_C(0^-) = 20 \text{ V} \\
 & \text{KVL: } -\frac{di_L}{dt} + 8 + 0.2 i_L - v_C + 1.8 i_L = 0 \\
 & v_C = \frac{1}{C} \int_0^t i_L(t) dt + v_C(0)
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{di_L}{dt} + 8 + 0.2 i_L - v_C + 1.8 i_L = 0 \\
 & v_C = \frac{1}{C} \int_0^t i_L(t) dt + v_C(0)
 \end{aligned}$$

Taking $\frac{d}{dt}$ on both sides \Rightarrow

$$\begin{aligned}
 & 5 \times 10^{-3} \frac{d^2 i_L}{dt^2} + (0.2 + 1.8) \frac{di_L}{dt} + \frac{1}{5 \times 10^{-3}} i_L = 0 \\
 & \frac{d^2 i_L}{dt^2} + \frac{2.0}{5 \times 10^{-3}} \frac{di_L}{dt} + \frac{1}{(5 \times 10^{-3})(5 \times 10^{-3})} i_L = 0
 \end{aligned}$$

$$\lambda = \frac{R}{2L} = \frac{2.0}{2 \times 5 \times 10^{-3}} = 200$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{5 \times 10^{-3}} = 200 \text{ rad/s}$$

$$\frac{d^2 i_L}{dt^2} + \frac{2.0}{5 \times 10^{-3}} \frac{di_L}{dt} + \frac{1}{(5 \times 10^{-3})(5 \times 10^{-3})} i_L = 0$$

$\lambda > \omega_0$ over damped

$$i_L(t) = A_1 e^{rt} + A_2 e^{\omega t} + v_C(\infty)$$

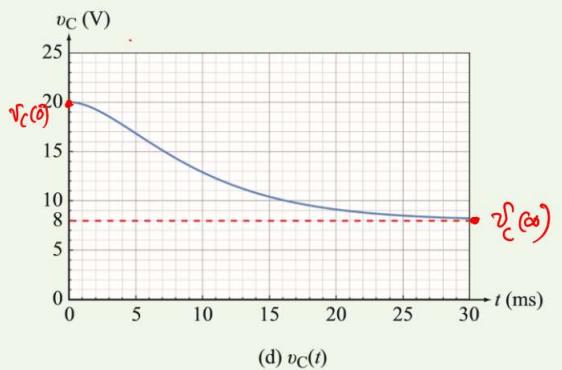
$$\begin{aligned}
s_1 &= -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\
&= -201 + \sqrt{(201)^2 - (200)^2} = -181 \text{ Np/s}, \\
s_2 &= -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -221 \text{ Np/s}, \\
A_1 &= \frac{\frac{1}{C} i_C(0) - s_2 [v_C(0) - v_C(\infty)]}{s_1 - s_2} \\
&= \frac{0 + 221[20 - 8]}{-181 + 221} = 66.3, \\
A_2 &= \frac{\frac{1}{C} i_C(0) - s_1 [v_C(0) - v_C(\infty)]}{s_2 - s_1} \\
&= \frac{0 + 181[20 - 8]}{-221 + 181} = -54.3.
\end{aligned}$$

Inserting the values of s_1, s_2, A_1, A_2 , and $v_C(\infty)$ in Eq. (6.16) leads to

$$\tau_1 = \frac{1}{181} \quad \tau_2 = \frac{1}{221}$$

$$v_C(t) = (66.3e^{-181t} - 54.3e^{-221t} + 8) \text{ V} \quad \text{for } t \geq 0.$$

$$\begin{aligned}
v_C(0) &= 66.3 - 54.3 + 8 = 20 [\text{V}] \\
v_C(\infty) &= 0 - 0 + 8 = 8 [\text{V}]
\end{aligned}$$



6-4 Series RLC Critically Damped Response ($\alpha = \omega_0$)

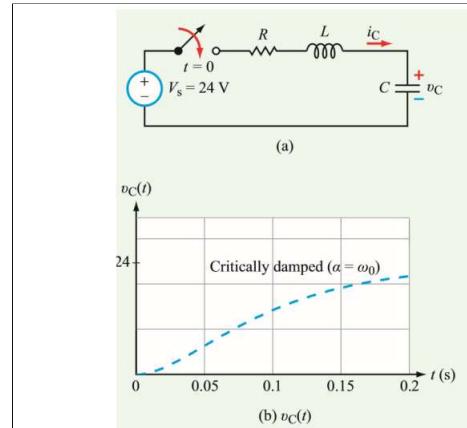
The critically damped response is the fastest response the circuit can exhibit, without oscillation, between initial and final conditions. ▶

When

$$\begin{aligned}
\omega &= \frac{R}{2L} = \omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow R = 2L \frac{1}{\sqrt{LC}} \\
&= 2\sqrt{\frac{L}{C}}
\end{aligned}$$

$$R = 2\sqrt{\frac{L}{C}} \quad (\text{critically damped}), \quad (6.23)$$

$$\begin{aligned}
v_C(t) &= A_1 e^{-\alpha t} + A_2 e^{-\alpha t} + v_C(\infty) \\
&= (A_1 + A_2)e^{-\alpha t} + v_C(\infty) = (A_3)e^{-\alpha t} + v_C(\infty),
\end{aligned} \tag{6.25}$$



Example 6-5: Critically Damped Response

Evaluate the response of the circuit in **Fig. 6-10(a)** for $t \geq 0$, given that the capacitor had no charge prior to $t = 0$ and $V_s = 24$ V, $R = 12 \Omega$, $L = 0.3$ H, and $C = 8.33 \text{ mF}$.

Solution: The parameters α and ω_0 are given by

$$\alpha = \frac{R}{2L} = \frac{12}{2 \times 0.3} = 20 \text{ Np/s},$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.3 \times 8.33 \times 10^{-3}}} = 20 \text{ rad/s.}$$

Hence, because $\alpha = \omega_0$, the response is critically damped and given by Eq. (6.26) as

$$v(t) = (B_1 + B_2 t)e^{-20t} + v_C(\infty).$$

The initial conditions at $t = 0$ are

$$v_C(0) = 0 \quad \text{and} \quad i_C(0) = 0,$$

and the final condition on v_C is

$$v_C(\infty) = V_s = 24 \text{ V.}$$

$$B_1 = v_C(0) - v_C(\infty) = -24 \text{ V,}$$

$$B_2 = \frac{1}{C} i_C(0) + \alpha[v_C(0) - v_C(\infty)] \\ = 0 + 20[0 - 24] = -480.$$

Hence,

$$v_C(t) = (B_1 + B_2 t)e^{-\alpha t} + v_C(\infty) \\ = [-(24 + 480t)e^{-20t} + 24] \text{ V,} \quad \text{for } t \geq 0.$$

6-5 Series RLC Underdamped Response ($\alpha < \omega_0$)

If $\alpha < \omega_0$, corresponding to

$$R < 2\sqrt{\frac{L}{C}} \quad (\text{underdamped}), \quad (6.29)$$

we introduce the **damped natural frequency** ω_d defined as

$$\omega_d^2 = \omega_0^2 - \alpha^2. \quad (6.30)$$

Since $\alpha < \omega_0$, it follows that $\omega_d > 0$. In terms of ω_d , the expressions for the roots s_1 and s_2 given by Eq. (6.18) become

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha + \sqrt{-\omega_d^2} = -\alpha + j\omega_d, \quad (6.31a)$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -\alpha - j\omega_d, \quad (6.31b)$$

where $j = \sqrt{-1}$. The fact that s_1 and s_2 are complex conjugates of one another will prove central to the form of the solution. Inserting the expressions for s_1 and s_2 into Eq. (6.16) gives

$$v_C(t) = A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t} + v_C(\infty). \quad (6.32)$$

$$v_C(t) = e^{-\alpha t}[D_1 \cos \omega_d t + D_2 \sin \omega_d t] + v_C(\infty)$$

(for $t \geq 0$) **(underdamped)**.

(6.35)

Example 6-6: Underdamped Response

Determine $v_C(t)$ for the circuit in **Fig. 6-11**, given that $V_s = 24$ V, $R = 12 \Omega$, $L = 0.3$ H, and $C = 0.72 \text{ mF}$. The circuit had been in steady state prior to moving the switch at $t = 0$.

Solution: For the specified values of R , L , and C ,

$$\alpha = \frac{R}{2L} = \frac{12}{2 \times 0.3} = 20 \text{ Np/s}$$

and

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.3 \times 0.72 \times 10^{-3}}} = 68 \text{ rad/s.}$$

Since $\alpha < \omega_0$, the voltage response is underdamped and given by Eq. (6.35) as

$$v_C(t) = e^{-\alpha t}[D_1 \cos \omega_d t + D_2 \sin \omega_d t] + v_C(\infty),$$

with

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{(68)^2 - (20)^2} = 65 \text{ rad/s.}$$

Prior to $t = 0$, the circuit was in steady state, which means that the capacitor was fully charged at $V_s = 24$ V and acting like an open circuit. Hence, $v_C(0^-) = 24$ V and $i_C(0^-) = 0$.

$$D_1 = v_C(0) - v_C(\infty), \quad (6.37a)$$

$$D_2 = \frac{\frac{1}{C} i_C(0) + \alpha[v_C(0) - v_C(\infty)]}{\omega_d}. \quad (6.37b)$$

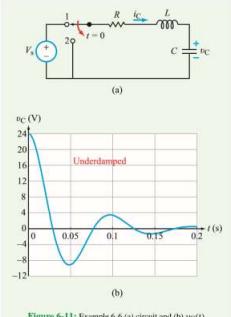


Figure 6-11: Example 6-6 (a) circuit and (b) $v_C(t)$.

Since both v_C across C and i_L through L cannot change instantaneously,

$$v_C(0) = 24 \text{ V.}$$

$$i_C(0) = i_L(0) = i_L(0^-) = 0.$$

After $t = 0$, the closed RLC circuit will no longer have any active sources, allowing the capacitor to dissipate all its energy in the resistor. Hence, as $t \rightarrow \infty$, $v_C(\infty) = 0$. Using these initial and final values in the appropriate expressions for D_1 and D_2 in Eq. (6.37) leads to $D_1 = 24 \text{ V}$, $D_2 = 7.4 \text{ V}$, and

$$v_C(t) = e^{-20t} [24 \cos 65t + 7.4 \sin 65t] \text{ V,} \quad \text{for } t \geq 0.$$

Figure 6-11(b) shows a time plot of $v_C(t)$, which exhibits an exponential decay (due to e^{-20t}) in combination with the oscillatory behavior associated with the sine and cosine functions.