

ECE101F19 Lecture 14, Nov. 14, 2019

HW #8 for Quiz 8 on Nov. 21

Quiz 9 will be given on Dec 3
and covers both HW9 & HW10

- | | |
|-------------|----------|
| 1) Prob 6.1 | 8) 6.18 |
| 2) 6.3 | 9) 6.22 |
| 3) 6.7 | 10) 6.28 |
| 4) 6.10 | |
| 5) 6.12 | |
| 6) 6.14 | |
| 7) 6.16 | |

$$\text{Qz 6 Avg} = 9.5 \quad \sigma = 1.5$$

Combining In-Series Capacitors

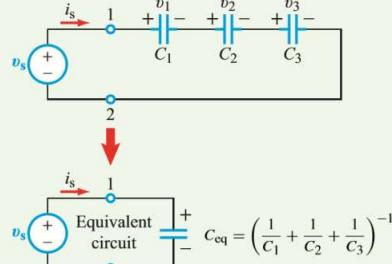


Figure 5-16: Capacitors in series.

Combining In-Parallel Capacitors

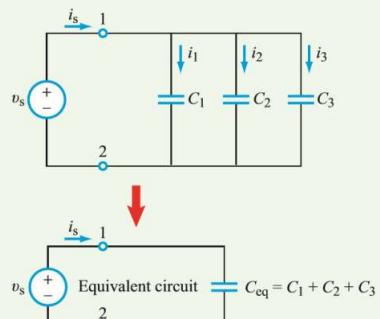


Figure 5-17: Capacitors in parallel.

Voltage Division

$$(a) v_1 = \left(\frac{R_1}{R_1 + R_2} \right) v_s \quad (b) v_1 = \left(\frac{C_2}{C_1 + C_2} \right) v_s$$

$$v_2 = \left(\frac{R_2}{R_1 + R_2} \right) v_s \quad v_2 = \left(\frac{C_1}{C_1 + C_2} \right) v_s$$

Figure 5-19: Voltage-division rules for (a) in-series resistors and (b) in-series capacitors.

In general $E = \frac{V_C}{C}$

$$RC \frac{dV_C}{dt} + V_C = E$$

$$\frac{dV_C}{dt} + \frac{1}{RC} V_C = \frac{E}{RC} \quad \alpha = \frac{1}{RC}$$

$$\frac{dV_C}{dt} + \alpha V_C = b$$

$$V_C(t) = V_C(\infty) + (V_C(0) - V_C(\infty)) e^{-\frac{t}{RC}}$$

$$E = R i_L + L \frac{di_L}{dt} + \frac{1}{C} V_C$$

$$\frac{di_L}{dt} + \frac{R}{L} i_L = \frac{E}{L}$$

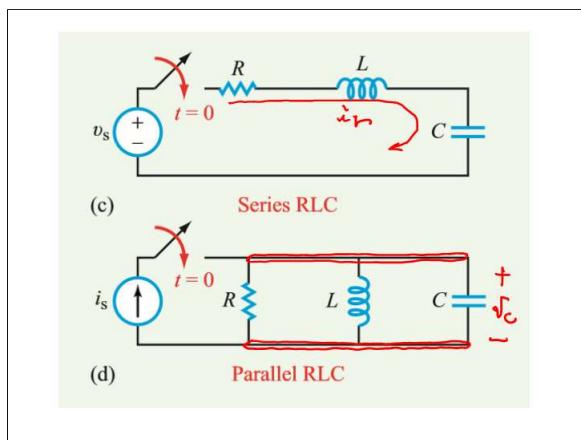
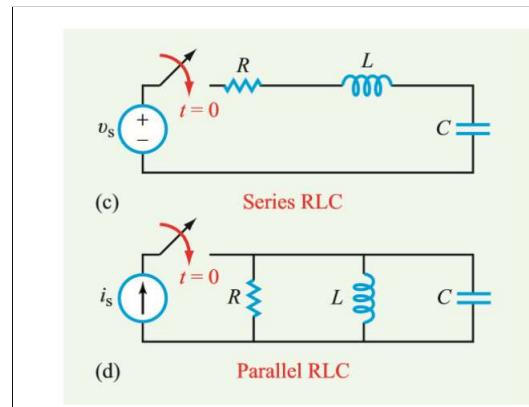
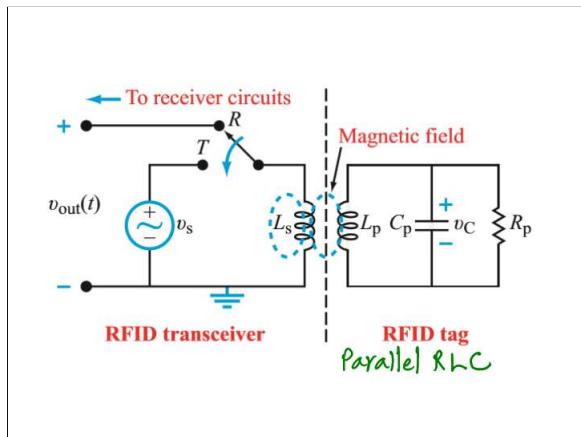
$$i_L(t) = i_L(0) + (i_L(0) - i_L(\infty)) e^{-at}$$

$$V_C(t) = \frac{1}{C} \int i_L(t) dt + C_0$$

$$V_C(t) = \frac{1}{C} \int [i_L(0) + (i_L(0) - i_L(\infty)) e^{-at}] dt + C_0$$

$$V_C(t) = \frac{1}{C} \left[i_L(0)t + (i_L(0) - i_L(\infty)) \frac{1 - e^{-at}}{a} \right] + C_0$$

$$V_C(t) = \frac{1}{C} i_L(0)t + \frac{1}{C} (i_L(0) - i_L(\infty)) \frac{1 - e^{-at}}{a} + C_0$$



Solving for $i_L(t)$:

$$v_s = R i_L + L \frac{di_L}{dt} + \frac{1}{C} \int i_L dt$$

$$\text{i)} \frac{d}{dt} \Rightarrow \frac{dv_s}{dt} = R \frac{di_L}{dt} + L \frac{d^2 i_L}{dt^2} + \frac{1}{C} i_L$$

$$\text{ii)} \div L \Rightarrow \frac{d^2 i_L}{dt^2} + \frac{R}{L} \frac{di_L}{dt} + \frac{1}{LC} i_L = \frac{1}{L} \frac{dv_s}{dt}$$

Solving for $v_C(t)$:

$$v_s = R C \frac{dv_C}{dt} + L \frac{1}{dt} \left(C \frac{dv_C}{dt} \right) + v_C$$

$$\div LC \Rightarrow \frac{d^2 v_C}{dt^2} + \frac{1}{L} \frac{dv_C}{dt} + \frac{1}{LC} v_C = \frac{1}{LC} \frac{dv_s}{dt}$$

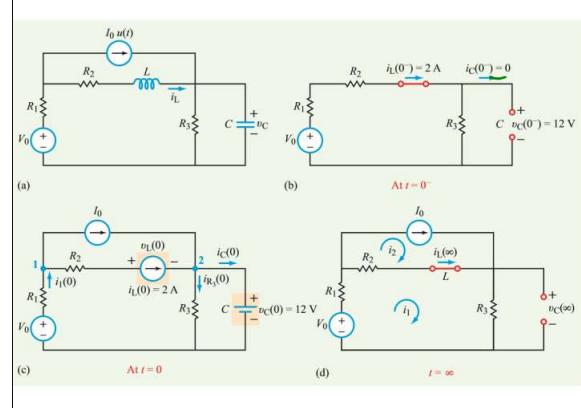
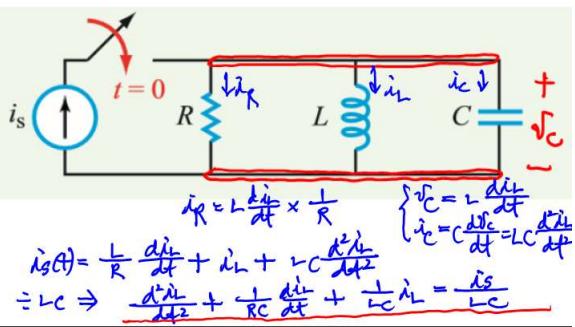
Solving for $v_C(t)$:

$$i_s = \frac{v_s}{R} + \frac{1}{L} \int v_C dt + C \frac{dv_C}{dt}$$

$$\text{i)} \frac{1}{L} \rightarrow \frac{dv_C}{dt} = \frac{1}{R} \frac{dv_s}{dt} + \frac{1}{L} v_C + C \frac{dv_C}{dt}$$

$$\text{ii)} \div C \rightarrow \frac{1}{C} \frac{dv_C}{dt} = \frac{1}{R} \frac{dv_s}{dt} + \frac{1}{RC} v_C + \frac{1}{LC} v_C$$

Solving for $i_L(t)$:



Initial conditions: $v_C(0^-) = 0$, $i_L(0^-) = 2 \text{ A}$, $i_C(0^-) = 0$.

Equation: $v_C(0) = \frac{1}{C} \int_{0^-}^{0^+} i_C dt$

Substitution: $v_C(0) = \frac{1}{C} \int_{0^-}^{0^+} i_C dt + v_C(0^-)$

Final result: $\boxed{v_C(0) = v_C(0^-)}$

Initial conditions: $v_C(0^-) = 0$, $i_L(0^-) = 2 \text{ A}$, $i_C(0^-) = 0$.

Equation: $i_L(0) = \frac{1}{L} \int_{0^-}^{0^+} v_L dt$

Substitution: $i_L(0) = i_L(0^-)$

Final result: $\boxed{i_L(0) = i_L(0^-)}$

Exercise 6-1: For the circuit in Fig. E6.1, determine $v_C(0)$, $i_L(0)$, $v_L(0)$, $i_C(0)$, $v_C(\infty)$, and $i_L(\infty)$.

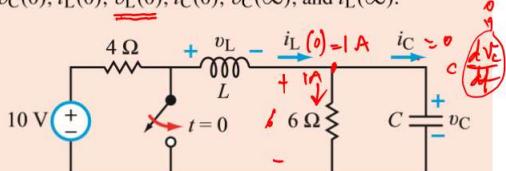


Figure E6.1

Answer: $v_C(0) = 6 \text{ V}$, $i_L(0) = 1 \text{ A}$, $v_L(0) = -6 \text{ V}$, $i_C(0) = 0$, $v_C(\infty) = 0$, $i_L(\infty) = 0$. (See CAD)

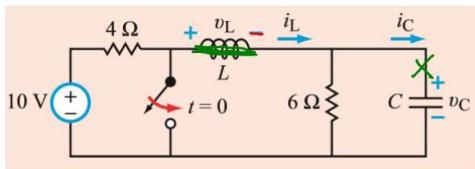
Initial conditions: $i_L(0) = 1 \text{ A}$, $v_C(0) = 0$.

Equation: $C \frac{dv_C}{dt} = i_C$

Equation: $C \frac{dv_C}{dt} = C \frac{(2-5)}{0-(0)} \propto$

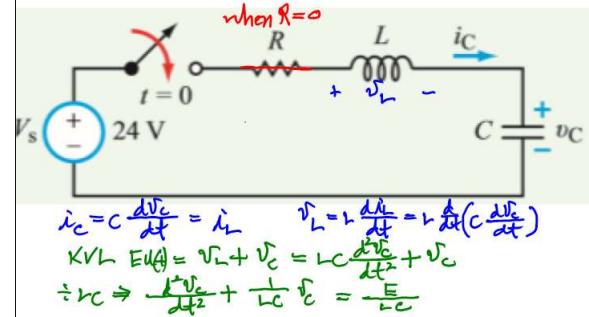
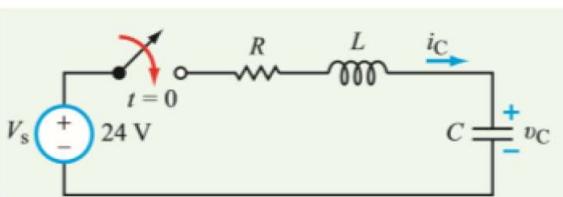
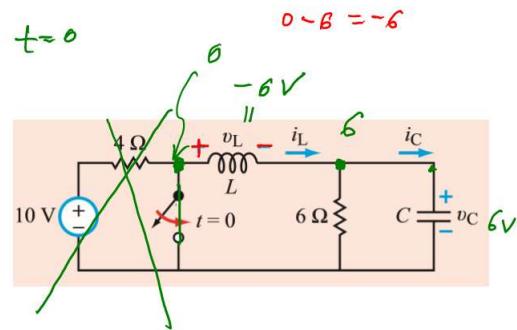
Final result: $v_C(0) = 2.5 \text{ V}$

At $t=0-$



$$i_L(0-) = 1 \text{ A} \quad v_L(0-) = 0 \text{ V}$$

$$v_C(0-) = 6 \text{ V}$$



$$i_C = C \frac{dv_C}{dt} = i_L \quad v_L = L \frac{di_L}{dt} = L \frac{di_C}{dt} = L \left(C \frac{dv_C}{dt} \right)$$

$$\text{KVL } E(t) = v_L + v_C = LC \frac{dv_C}{dt} + v_C$$

$$\frac{1}{LC} \Rightarrow \frac{1}{L^2} v_C + \frac{1}{LC} v_C = \frac{E}{LC}$$

$\downarrow \omega$

$$s^2 v_C(s) - s v_C(0) - v_C'(0) + \frac{1}{LC} v_C(s) = \frac{E}{LC} \frac{1}{s}$$

$$(s^2 + \frac{1}{LC}) v_C(s) = \frac{E}{LC} \frac{1}{s} + s v_C(0) + v_C'(0)$$

$$v_C(s) = \frac{E}{LC} \frac{1}{s(s+\frac{1}{\omega_0^2})} + \frac{s v_C(0) + v_C'(0)}{s^2 + \frac{1}{LC}}$$

$$= \frac{E}{LC} \left[\frac{A}{s} + \frac{Bs+C}{s^2 + \frac{1}{LC}} + \frac{s v_C(0) + v_C'(0)}{s^2 + \frac{1}{LC}} \right]$$

If $v_C(0) = 0$, $v_C(s) = \frac{E}{LC} \left[\frac{A}{s} + \frac{Bs+C}{s^2 + \frac{1}{LC}} \right]$

$$v_C'(0) = 0$$

$$\text{Let } \frac{1}{LC} = \omega_0^2, \text{ then } v_C(s) = \omega_0^2 E \frac{As^2 + Aw_0^2 + Bs^2 + Cs}{s(s + \omega_0^2)}$$

$$\Rightarrow A + B = 0, A\omega_0^2 = 1, C = 0$$

$$A = \frac{1}{\omega_0^2}, B = -\frac{1}{\omega_0^2}$$

$$v_C(s) = \omega_0^2 E \left[\frac{1}{\omega_0^2 s} + \frac{1}{s^2 + \omega_0^2} + \left(-\frac{1}{\omega_0^2 s} \right) \frac{s}{s^2 + \omega_0^2} \right]$$

Laplace Transform Pairs	
$f(t)$	$F(s) = \mathcal{L}[f(t)]$
$\delta(t)$	1
$\delta(t-T)$	e^{-Ts}
1 or $u(t)$	$\frac{1}{s}$
$e^{-at} u(t)$	$\frac{1}{s+a}$
$e^{-at} u(t-T)$	$\frac{e^{-aT}}{s+a}$
$t u(t)$	$\frac{1}{s^2}$
$(t-T) u(t-T)$	$\frac{e^{-Ts}}{s^2}$
$t^2 u(t)$	$\frac{2}{s^3}$
$t e^{-at} u(t)$	$\frac{1}{(s+a)^2}$
$t^2 e^{-at} u(t)$	$\frac{(s+a)^3}{(s+a)^2}$
$t^{n-1} e^{-at} u(t)$	$\frac{(n-1)!}{(s+a)^n}$
$\sin \omega t u(t)$	$\frac{\omega}{s^2 + \omega^2}$
$\sin(\omega t + \theta) u(t)$	$\frac{s \sin \theta + \omega \cos \theta}{s^2 + \omega^2}$
$\cos \omega t u(t)$	$\frac{s \cos \theta}{s^2 + \omega^2}$

Recall that we set $R=0$ (lossless)

$$\omega_0^2 = \frac{1}{LC} \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$

resonant frequency

$$V_C(s) = E \left[\frac{1}{s} - \frac{s}{s^2 + \omega_0^2} \right]$$

$$V_C(t) = E \left(1 - \cos \omega_0 t \right), t \geq 0$$

If $E=0$, but $V_C(0) = V_{CO}$, $V_C'(0)=0$, then

$$V_C(s) = \frac{s V_C(0) + V_C'(0)}{s^2 + \omega_0^2} = \frac{s V_{CO} + 0}{s^2 + \omega_0^2}$$

$$\downarrow z^{-1}$$

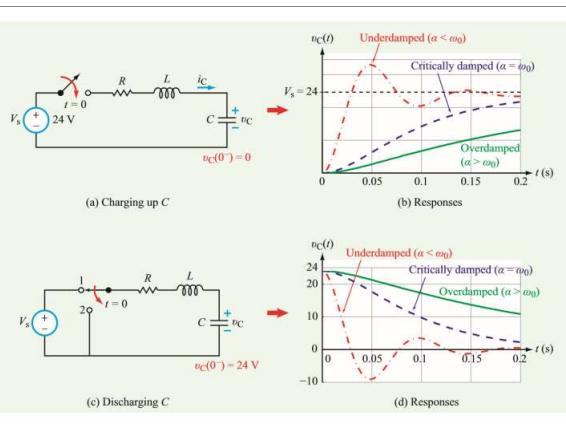
$$V_C(t) = V_{CO} \cos \omega_0 t, t \geq 0$$

If $R=0$, $d=0$ (no damping)

damping coefficient $\alpha = \frac{R}{2L}$ (Np/s), (6.1a)
[neper/s]

resonant frequency $\omega_0 = \frac{1}{\sqrt{LC}}$ (rad/s). (6.1b)
(series RLC) angular freq.

Overdamped response	$\alpha > \omega_0$,
Critically damped response	$\alpha = \omega_0$,
Underdamped response	$\alpha < \omega_0$.



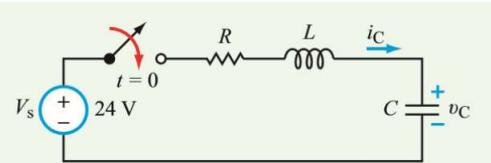


Figure 6-7: Series RLC circuit connected to a source V_s at $t = 0$. In general, the capacitor may have had an initial charge on it at $t = 0^-$, with a corresponding initial voltage $v_C(0^-)$.

6-3.1 Differential Equation

For the circuit in **Fig. 6-7**, the KVL loop equation for $t \geq 0$ (after closing the switch) is

$$t > 0 \quad R i_C + L \frac{di_C}{dt} + v_C = V_s \quad (\text{for } t \geq 0), \quad (6.2)$$

$$\frac{d^2v_C}{dt^2} + \frac{R}{L} \frac{dv_C}{dt} + \frac{1}{LC} v_C = \frac{V_s}{LC}. \quad (6.4)$$

For convenience, we rewrite Eq. (6.4) in the abbreviated form

$$v_C'' + av_C' + bv_C = c, \quad (6.5)$$

where

$$a = \frac{R}{L}, \quad b = \frac{1}{LC}, \quad c = \frac{V_s}{LC}. \quad (6.6)$$

6-3.2 Solution of Differential Equation

The general solution of the second-order differential equation given by Eq. (6.5) consists of two components:

$$v_C(t) = v_{tr}(t) + v_{ss}(t), \quad (6.7)$$

where $v_{tr}(t)$ is the transient (also called homogeneous) solution of Eq. (6.5) or the natural response of the RLC circuit and $v_{ss}(t)$ is the steady-state solution (also called particular solution). The transient solution is the solution of Eq. (6.5) under source-free conditions; i.e., with $V_s = 0$, which means that $c = V_s/LC$ also is zero. Thus $v_{tr}(t)$ is the solution of

$$v_{tr}'' + av_{tr}' + bv_{tr} = 0 \quad (\text{source-free}). \quad (6.8)$$

$$s = -\frac{a}{2} + j\omega_{\text{mag}} \quad v_{tr}(t) = A e^{st}, \quad (6.11)$$

where A and s are constants to be determined later. To ascertain that Eq. (6.11) is indeed a viable solution of Eq. (6.8), we insert the proposed expression for $v_{tr}(t)$ and its first and second derivatives in Eq. (6.8). The result is

$$s^2 A e^{st} + as e^{st} + b A e^{st} = 0, \quad (6.12)$$

which simplifies to

$$s^2 + as + b = 0. \quad (6.13)$$

Hence, the proposed solution given by Eq. (6.11) is indeed an acceptable solution so long as Eq. (6.13) is satisfied.

The quadratic equation given by Eq. (6.13) is known as the characteristic equation of the differential equation. It has two roots:

$$s_1 = -\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b}, \quad (6.14a)$$

$$s_2 = -\frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b}. \quad (6.14b)$$

The existence of two distinct roots implies that Eq. (6.8) has two viable solutions, one in terms of $e^{s_1 t}$ and another in terms of $e^{s_2 t}$. Hence, we should generalize the form of our solution to

$$v_{tr}(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{for } t \geq 0, \quad (6.15)$$

where constants A_1 and A_2 are to be determined shortly.

Inserting Eq. (6.15) into Eq. (6.10) leads to

$$v_C(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + v_C(\infty). \quad (6.16)$$

6-3.3 Invoking Initial Conditions

To determine the values of constants A_1 and A_2 in Eq. (6.16), we need to **invoke initial conditions**, which means that we need to use information available to us about the values of v_C and its time derivative i_C' , both at $t = 0$. Since

$$i_C(t) = C \frac{dv_C}{dt} = C v'(t), \quad (6.19)$$

the second requirement is equivalent to needing to know $i_C(0)$. At $t = 0$, Eq. (6.16) simplifies to

$$v_C(0) = A_1 + A_2 + v_C(\infty), \quad (6.20)$$

and

$$\begin{aligned} i_C(0) &= C \left. \frac{dv_C}{dt} \right|_{t=0} = C(s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}) \Big|_{t=0} \\ &= C(s_1 A_1 + s_2 A_2). \end{aligned} \quad (6.21)$$

Simultaneous solution of Eqs. (6.20) and (6.21) for A_1 and A_2 gives

$$A_1 = \frac{\frac{1}{C} i_C(0) - s_2 [v_C(0) - v_C(\infty)]}{s_1 - s_2}, \quad (6.22a)$$

$$A_2 = \frac{\frac{1}{C} i_C(0) - s_1 [v_C(0) - v_C(\infty)]}{s_2 - s_1}. \quad (6.22b)$$

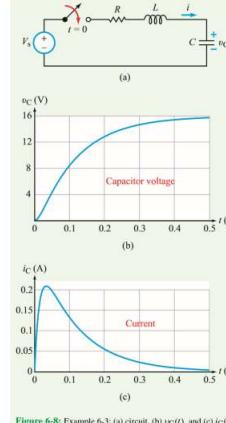


Figure 6-8: Example 6-3: (a) circuit, (b) $v_C(t)$, and (c) $i_C(t)$

Table 6-1: Step response of RLC circuits for $t \geq 0$.

Series RLC	Parallel RLC
Total Response	Total Response
Overshoot ($\alpha > \omega_0$)	Overshoot ($\alpha > \omega_0$)
$v_C(t) = A_1 e^{\alpha t} + A_2 e^{\alpha t} + v_C(\infty)$	$i_L(t) = A_1 e^{\alpha t} + A_2 e^{\alpha t} + i_L(\infty)$
$A_1 = \frac{\frac{1}{C} i_C(0) - s_1 [v_C(0) - v_C(\infty)]}{s_1 - s_2}$	$A_1 = \frac{\frac{1}{C} v_C(0) - s_2 [i_L(0) - i_L(\infty)]}{s_1 - s_2}$
$A_2 = \left[\frac{\frac{1}{C} i_C(0) - s_1 [v_C(0) - v_C(\infty)]}{s_2 - s_1} \right]$	$A_2 = \left[\frac{\frac{1}{C} v_C(0) - s_1 [i_L(0) - i_L(\infty)]}{s_2 - s_1} \right]$
Critically Damped ($\alpha = \omega_0$)	Critically Damped ($\alpha = \omega_0$)
$v_C(t) = (B_1 + B_2 t)e^{-\alpha t} + v_C(\infty)$	$i_L(t) = (B_1 + B_2 t)e^{-\alpha t} + i_L(\infty)$
$B_1 = v_C(0) - v_C(\infty)$	$B_1 = i_L(0) - i_L(\infty)$
$B_2 = \frac{1}{C} i_C(0) + \alpha(v_C(0) - v_C(\infty))$	$B_2 = \frac{1}{C} v_C(0) + \alpha(i_L(0) - i_L(\infty))$
Underdamped ($\alpha < \omega_0$)	Underdamped ($\alpha < \omega_0$)
$v_C(t) = e^{-\alpha t}(D_1 \cos \omega_0 t + D_2 \sin \omega_0 t) + v_C(\infty)$	$i_L(t) = e^{-\alpha t}(D_1 \cos \omega_0 t + D_2 \sin \omega_0 t) + i_L(\infty)$
$D_1 = v_C(0) - v_C(\infty)$	$D_1 = i_L(0) - i_L(\infty)$
$D_2 = \frac{1}{C} i_C(0) + \alpha(v_C(0) - v_C(\infty))$	$D_2 = \frac{1}{C} v_C(0) + \alpha(i_L(0) - i_L(\infty))$
Auxiliary Relations	
$\alpha = \sqrt{\frac{R^2}{4L} + \frac{1}{LC}}$	
$\omega_0 = \sqrt{\frac{1}{LC} - \alpha^2}$	
$s_1 = -\alpha + \sqrt{\omega_0^2 - \alpha^2}$	
$s_2 = -\alpha - \sqrt{\omega_0^2 - \alpha^2}$	