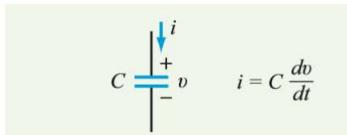


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$$\alpha = 5 \quad \text{Avg} = 6.82 \\ \alpha = 2.48$$



$$i(t) = C \frac{dv(t)}{dt}$$

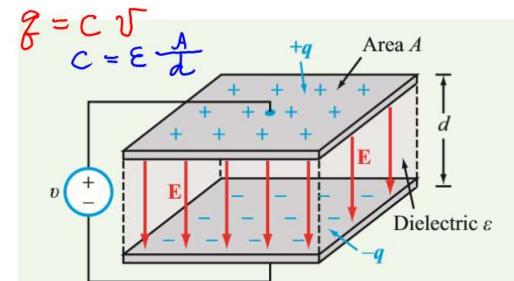
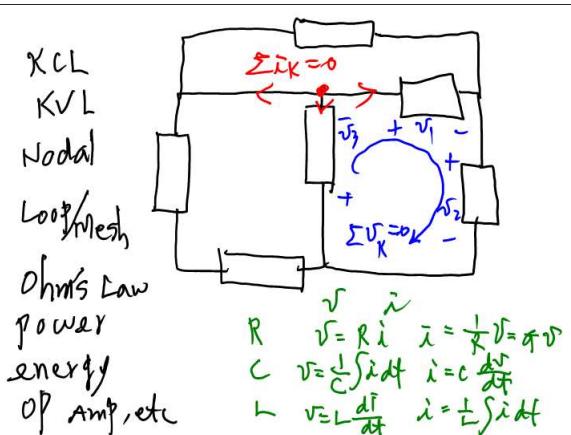
$$\frac{d^2v(t)}{dt^2} = \frac{1}{C} i(t)$$

$$\int \frac{d^2v(t)}{dt^2} dt = \frac{1}{C} \int i(t) dt$$

$$\int_{-\infty}^t \frac{d^2v(t)}{dt^2} dt = \frac{1}{C} \int_{-\infty}^t i(t) dt$$

$$v(t) - v(\infty) = \frac{1}{C} \left[ \int_{-\infty}^0 i(t) dt + \int_{0}^t i(t) dt \right]$$

$$\Rightarrow v(t) = \underbrace{\frac{1}{C} f(a)}_{= v(\infty)} + \frac{1}{C} \int_{0}^t i(t) dt$$

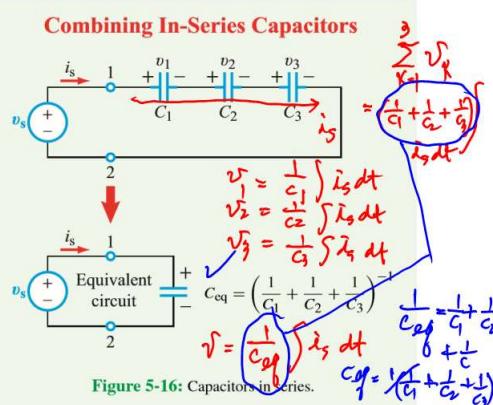


**Figure 5-11:** Parallel-plate capacitor with plates of area  $A$ , separated by a distance  $d$ , and filled with an insulating dielectric material of permittivity  $\epsilon$ .

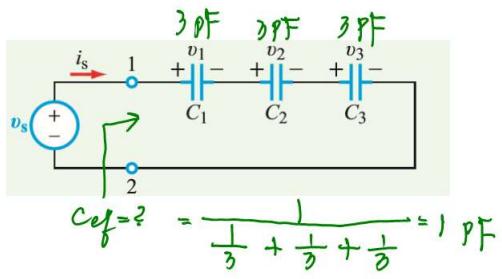
$$\epsilon_0 = 8.854 \times 10^{-12} \text{ F/cm}$$

**Table 5-2:** Relative electrical permittivity of common insulators:  $\epsilon_r = \epsilon/\epsilon_0$  and  $\epsilon_0 = 8.854 \times 10^{-12} \text{ F/m}$ .

Material	Relative Permittivity $\epsilon_r$
Air (at sea level)	1.0006
Teflon	2.1
Polystyrene	2.6
Paper	2-4
Glass	4.5-10
Quartz	3.8-5
Bakelite	5
Mica	5.4-6
Porcelain oxide	5.7 3.9



**Figure 5-16:** Capacitors in series.



$$C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} = \frac{C_1 C_2}{C_1 + C_2}$$

$$C_{eq} = \frac{1}{\frac{1}{5 \times 10^{-15}} + \frac{1}{10 \times 10^{-15}}} = \frac{1}{\frac{1}{5} + \frac{1}{10}} = \frac{10}{3} \text{ pF}$$

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

### Combining In-Parallel Capacitors

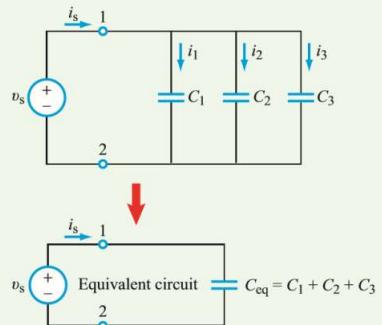
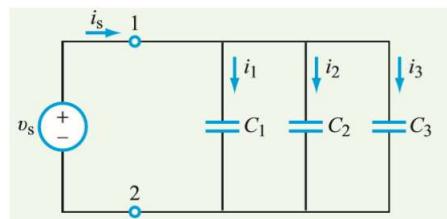


Figure 5-17: Capacitors in parallel.

$$i_s = i_1 + i_2 + i_3 = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt} = (C_1 + C_2 + C_3) \frac{dv}{dt}$$

$$i_s = C_{eq} \frac{dv}{dt}$$



**Exercise 5-9:** Determine  $C_{eq}$  and  $v_{eq}(0)$  at terminals (a, b) for the circuit in Fig. E5.9 given that  $C_1 = 6 \mu\text{F}$ ,  $C_2 = 4 \mu\text{F}$ ,  $C_3 = 8 \mu\text{F}$ , and the initial voltages on the three capacitors are  $v_1(0) = 5 \text{ V}$  and  $v_2(0) = v_3(0) = 10 \text{ V}$ , respectively.

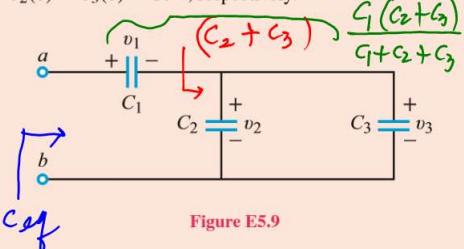


Figure E5.9

Answer:  $C_{eq} = 4 \mu\text{F}$ ,  $v_{eq}(0) = 15 \text{ V}$ . (See CAD)

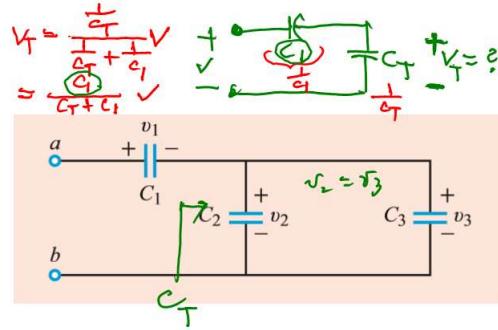
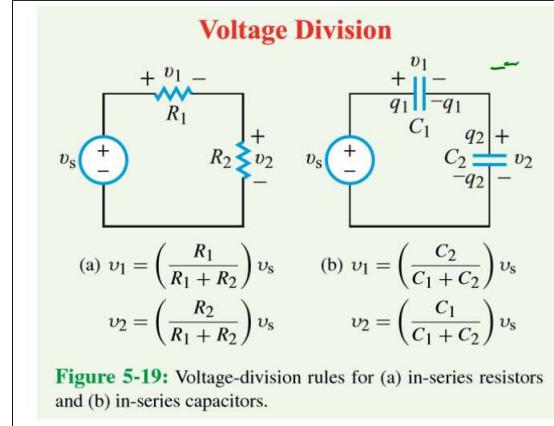
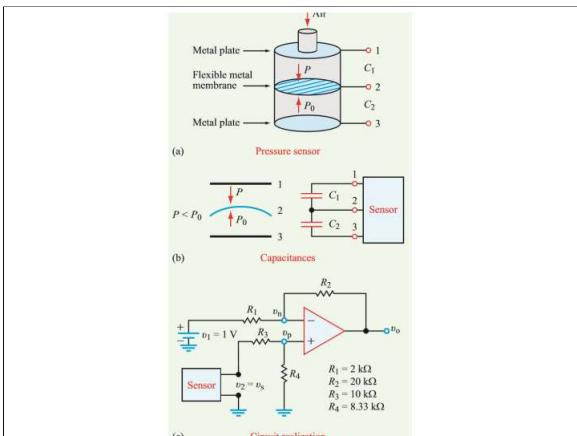



Figure 5-19: Voltage-division rules for (a) in-series resistors and (b) in-series capacitors.



Prob: Find  $v_c(t)$

$i_c(t) = C \frac{dv_c}{dt}$

$KVL: E_u(t) = R i_c(t) + v_c(t)$

$E_u(t) = R C \frac{dv_c}{dt} + v_c(t)$

For  $t > 0$   $E = R C \frac{dv_c}{dt} + v_c$

In steady state, that is as  $t \rightarrow \infty$

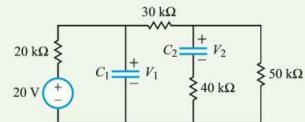
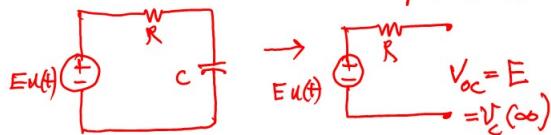
$$v_C(t) = v_C(\infty) \text{ constant}$$

$$i_C(t) = C \frac{dv_C(t)}{dt}$$

As  $t \rightarrow \infty$

$$i_C(\infty) = C \frac{dv_C(\infty)}{dt} = C \times 0 = 0$$

open circuit



(a) Original circuit

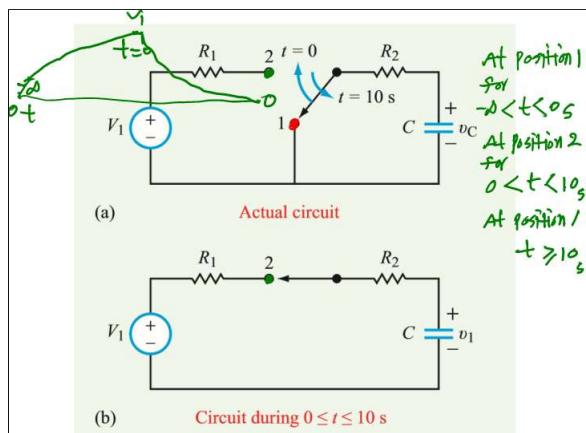
As  $t \rightarrow \infty$



(b) Equivalent circuit

$$\begin{aligned} R_{\text{loop}} &= 100\text{k} \\ V_2 &= \frac{20}{100\text{k}} 50\text{k} \\ &= 10\text{V} \\ V_1 &= \frac{20}{100\text{k}} (80\text{k}) \\ &= 16\text{V} \end{aligned}$$

Figure 5-15: Under dc conditions, capacitors behave like open circuits.



At position 1  
for  
 $0 < t < 0.5$   
At position 2  
for  
 $0 < t < 10\text{s}$   
At position 1  
 $t \geq 10\text{s}$

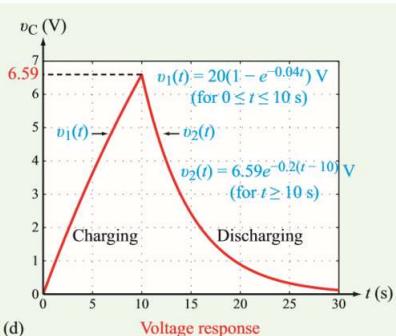
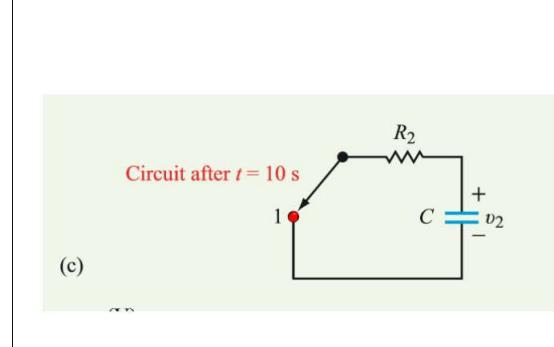


Figure 5-34: After having been in position 1 for a long time, the switch is moved to position 2 at  $t = 0$  and then returned to position 1 at  $t = 10\text{s}$  (Example 5-11).

In general

$$RC \frac{dv_C}{dt} + v_C = E$$

$$\frac{dv_C}{dt} + \underbrace{\frac{1}{RC} v_C}_{\alpha} = \frac{E}{RC} \quad \alpha = \frac{1}{RC}$$

$$\boxed{\frac{dv_C}{dt} + \alpha v_C = b}$$

Multiply both sides by  $e^{\alpha t}$

$$e^{\alpha t} \frac{dv_C}{dt} + \alpha e^{\alpha t} v_C = b e^{\alpha t}$$

Integrating both sides

$$\int_0^t e^{\alpha t} \frac{dv_C}{dt} dt + \int_0^t \alpha e^{\alpha t} v_C dt = \int_0^t b e^{\alpha t} dt$$

$$\begin{aligned}
 & \Rightarrow \int_0^t \frac{d}{dt} (V_c e^{at}) dt = \int_0^t b e^{at} dt \\
 & V_c e^{at} - V_c(0) = \int_0^t \frac{d}{dt} (b e^{at}) dt \\
 & V_c e^{at} - V_c(0) = \frac{b}{a} (e^{at} - 1) \\
 & V_c e^{at} = V_c(0) + \frac{b}{a} (e^{at} - 1) \\
 & V_c(t) = V_c(0) e^{-at} + \frac{b}{a} (1 - e^{-at}) \\
 & = \frac{b}{a} + (V_c(0) - \frac{b}{a}) e^{-at} \\
 & = V_c(0) + (V_c(0) - V_c(\infty)) e^{-at} \quad (5.96)
 \end{aligned}$$

Recall  $\alpha = \frac{1}{RC}$

$$b = \frac{E}{RC}$$

$$\frac{b}{\alpha} = \frac{\frac{E}{RC}}{\frac{1}{RC}} = E = V_c(\infty)$$

$V_c(0)$  initial voltage at  $t = 0^-$

Revisit L.e.

$$\frac{dV_c}{dt} + \alpha V_c = b$$

Laplace transformation method:

$$\mathcal{L}\left[\frac{dV_c}{dt} + \alpha V_c = b\right] \Rightarrow [sV_c(s) - V_c(0)] + \alpha V_c(s) = \frac{b}{s}$$

$$\left\{ \begin{array}{l} \mathcal{L}\left[\frac{dV_c}{dt}\right] = sV_c(s) - V_c(0) \\ \mathcal{L}[aV_c(t)] = aV_c(s) \end{array} \right.$$

$$\left\{ \begin{array}{l} \mathcal{L}[b] = \frac{b}{s} \\ \Rightarrow (s+a)V_c(s) = \frac{b}{s} + V_c(0) \end{array} \right.$$

$$V_c(s) = \frac{b}{s(s+a)} + \frac{V_c(0)}{s+a}$$

## CHAPTER 12

### Circuit Analysis by Laplace Transform

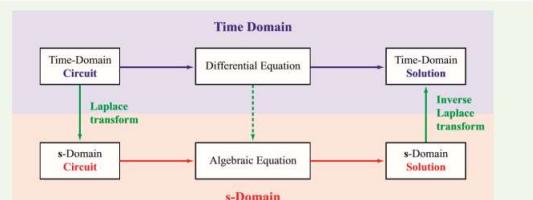


Figure 12-2: The top horizontal sequence involves solving a differential equation entirely in the time domain. The bottom horizontal sequence involves a much easier solution of a linear equation in the s-domain.

#### Solution Procedure: Laplace Transform

**Step 1:** The circuit is transformed to the Laplace domain—also known as the s-domain.

**Step 2:** In the s-domain, application of KVL and KCL yields a set of algebraic equations.

**Step 3:** The equations are solved for the variable of interest.

**Step 4:** The s-domain solution is transformed back to the time domain.

$$F(s) = \mathcal{L}[f(t)] = \int_{0^-}^{\infty} f(t) e^{-st} dt, \quad (12.10)$$

$$\begin{aligned} \text{e.g. } \mathcal{L}[e^{-at}] &= \int_{0^-}^{\infty} e^{-at} e^{-st} dt \\ &= \int_{0^-}^{\infty} e^{-(s+a)t} dt = \left[ -\frac{e^{-(s+a)t}}{s+a} \right]_{0^-}^{\infty} \\ &= -\frac{1}{s+a} (e^{\infty} - e^{(0)}) = \underline{\underline{-\frac{1}{s+a}}} \end{aligned}$$

$$\begin{aligned} \text{e.g. } \mathcal{L}[E] &= \int_{0^-}^{\infty} E e^{-st} dt \\ &= E \left[ -\frac{e^{-st}}{s} \right]_{0^-}^{\infty} \\ &= \frac{E}{s} (e^{\infty} - e^{(0)}) = \underline{\underline{\frac{E}{s}(0-1)}} \\ &= \underline{\underline{\frac{E}{s}}} \end{aligned}$$

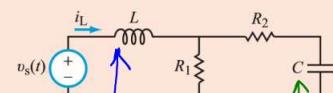
Table 12-1: Properties of the Laplace transform ( $f(t) = 0$  for  $t < 0^-$ ).

Property	$f(t)$	$F(s) = \mathcal{L}[f(t)]$
1. Multiplication by constant $K$	$K f(t)$	$\rightarrow K F(s)$
2. Linearity	$K_1 f_1(t) + K_2 f_2(t)$	$\rightarrow K_1 F_1(s) + K_2 F_2(s)$
3. Time scaling	$f(at), \quad a > 0$	$\rightarrow \frac{1}{a} F\left(\frac{s}{a}\right)$
4. Time shift	$f(t-T) u(t-T)$	$\rightarrow e^{-Ts} F(s), \quad T \geq 0$
5. Frequency shift	$e^{-at} f(t)$	$\rightarrow F(s+a)$
6. Time 1st derivative	$f' = \frac{df}{dt}$	$\rightarrow s F(s) - f(0^-)$
7. Time 2nd derivative	$f'' = \frac{d^2 f}{dt^2}$	$\rightarrow s^2 F(s) - s f(0^-) - f'(0^-)$
8. Time integral	$\int_0^t f(\tau) d\tau$	$\rightarrow \frac{1}{s} F(s)$
9. Frequency derivative	$t f(t)$	$\rightarrow \frac{d}{ds} F(s)$
10. Frequency integral	$\frac{f(t)}{t}$	$\rightarrow \int_s^\infty F(s') ds'$

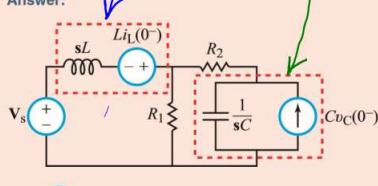
$$\begin{aligned} v_c(s) &= \frac{b}{s(s+a)} + \frac{v_c(\infty)}{s+a} \\ &= b \left[ \frac{\frac{1}{a}}{s} + \frac{-\frac{1}{a}}{s+a} \right] + \frac{v_c(\infty)}{s+a} \\ &= \frac{(\frac{b}{a})}{s} + \frac{v_c(\infty) - \frac{b}{a}}{s+a} \\ &\downarrow \mathcal{L}^{-1} \text{ Inverse Laplace Transform} \\ v_c(t) &= \frac{b}{a} + (v_c(\infty) - \frac{b}{a}) e^{-at} \\ &= v_c(\infty) + (v_c(\infty) - v_c(\infty)) e^{-at} \end{aligned}$$

$$\begin{aligned} \frac{b}{s(s+a)} &\Leftarrow \frac{A \frac{1}{a}}{s} + \frac{B \frac{-1}{a}}{s+a} \\ &= \frac{Aa + Aa + Bs}{s(s+a)} = \frac{b}{s(s+a)} \\ \begin{matrix} A+B=0 \\ Aa=b \end{matrix} &\quad \begin{matrix} A=-B \\ A=\frac{b}{a} \end{matrix} \\ \begin{matrix} E \\ \frac{b}{a} \end{matrix} &\quad \begin{matrix} B=-\frac{b}{a} \end{matrix} \\ \frac{b}{a} - \frac{b}{a} e^{-at} & \\ \Leftarrow E(1 - e^{-at}) & \end{aligned}$$

Exercise 12-9: Convert the circuit in Fig. E12.9 into the s-domain.



Answer:



(See CAD)

$$i_C = C \frac{dV_C}{dt}$$

Taking  $\int$  on both sides

$$I_C(s) = C [s V_C(s) - v_C(\infty)]$$

$$= C s V_C(s) - C v_C(\infty)$$

Inductor case (+time invariant L)

$$v_L = L i_L$$

$$v_L(s) = L i_L(s)$$

$$\frac{dv_L}{dt} = v_L'(t) = \frac{d}{dt}(L i_L(t))$$

$$= L \frac{di_L}{dt}$$

$$v_L(s) = L(s I_L(s) - i_L(\infty))$$

$$\frac{E - V_C(s)}{R} = C s V_C(s) - C v_C(\infty)$$

$$\frac{E - V_C(s)}{R} = C s V_C(s) - C v_C(\infty)$$

$$\frac{E - V_C(s)}{R} = C s V_C(s) - C v_C(\infty)$$

$$\frac{E}{R} - \frac{V_C(s)}{R} = C s V_C(s) - C v_C(\infty)$$

$$\frac{E}{R} = (R C s + 1) V_C(s) - R C v_C(\infty)$$

$$\Rightarrow V_C(s) = \frac{\frac{E}{R} + R C v_C(\infty)}{R C s + 1}$$

$$= \frac{E}{R C s + 1} + \frac{R C v_C(\infty)}{R C s + 1}$$

$$= \frac{E}{R C s (s + \frac{1}{R C})} + \frac{R C v_C(\infty)}{R C (s + \frac{1}{R C})}$$

$$= \frac{E}{R C} \left( \frac{1}{s(s + \frac{1}{R C})} \right) + \frac{v_C(\infty)}{s + \frac{1}{R C}}$$

$$= \frac{E}{R C} \left( \frac{\frac{1}{R C}}{s} + \frac{-\frac{1}{R C}}{s + \frac{1}{R C}} \right) + \frac{v_C(\infty)}{s + \frac{1}{R C}}$$

$$= \frac{E}{s} + \frac{v_C(\infty) - E}{s + \frac{1}{R C}}$$

$$V_C(t) = \frac{E}{s} + \frac{(v_C(\infty) - E)}{s + \frac{1}{R C}} e^{-\frac{t}{R C}}$$

**Exercise 5-15:** Determine  $v_1(t)$  and  $v_2(t)$  for  $t \geq 0$ , given that in the circuit of Fig. E5.15  $C_1 = 6 \mu F$ ,  $C_2 = 3 \mu F$ ,  $R = 100 k\Omega$ , and neither capacitor had any charge prior to  $t = 0$ .

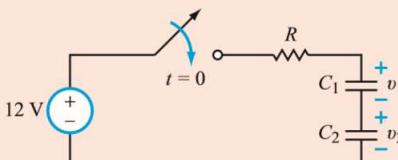
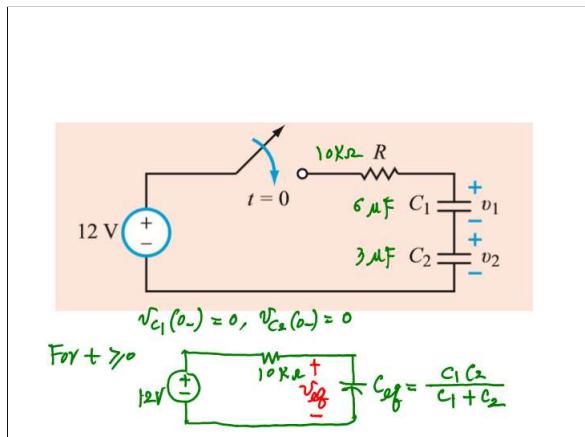


Figure E5.15

Answer:  $v_1(t) = 4(1 - e^{-5t})$  V, for  $t \geq 0$ ,  
 $v_2(t) = 8(1 - e^{-5t})$  V, for  $t \geq 0$ . (See CAD)

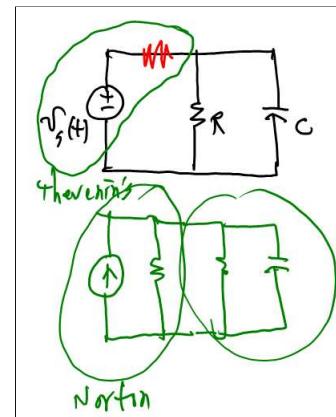


KVL:  $v_{12} = 10 \times e^t C_{ef} \frac{dv_{ef}}{dt} + v_{ef}$   
 $C_{ef} = \frac{C_1 C_2}{C_1 + C_2} = \frac{6\mu F \times 3\mu F}{6\mu F + 3\mu F} = 2\mu F$   
 $\Rightarrow v_{12} = [20 \times 10] \frac{dV_{ef}}{dt} + V_{ef}$   
 $\frac{dV_{ef}}{dt} + \left(\frac{1}{20 \times 10^{-3}}\right) V_{ef} = \frac{12}{20 \times 10^{-3}}$   
 $V_{ef}(t) = V_{ef}(0) + (V_{ef}(0) - V_{ef}(0)) e^{-\frac{t}{2 \times 10^{-3}}}$   
 $v_{12} = 12 - 12 e^{-\frac{500t}{1}}$

$i_{12}(t) = \frac{1}{C_1 + C_2} \int v_{12} dt = \frac{1}{6\mu F} \int 2\mu F \frac{dv_{ef}}{dt} dt$   
 $i_{12}(t) = 3\mu F \frac{dv_{ef}}{dt}$   
 $v_1(t) = \frac{1}{C_1} \int_{0-}^t i_{12}(t') dt' = \frac{1}{6\mu F} \int_{0-}^t 2\mu F \frac{dv_{ef}}{dt'} dt'$   
 $v_1(t) = \frac{1}{3} [v_{ef}(t) - v_{ef}(0)]$   
 $v_2(t) = \frac{1}{C_2} \int_{0-}^t i_{12}(t') dt' = \frac{1}{3\mu F} \int_{0-}^t 2\mu F \frac{dv_{ef}}{dt'} dt'$   
 $v_2(t) = \frac{1}{3} v_{ef}(t)$

$v_1 = \frac{C_2}{C_1 + C_2} v_{12}$   
 $v_2 = \frac{C_1}{C_1 + C_2} v_{12}$

$i_1 = \frac{1}{C_1} \int v_{12} dt = \frac{1}{C_1} (C_1 + C_2) i_{12}$   
 $i_1 = \frac{C_1}{C_1 + C_2} i_{12}$   
 $i_2 = \frac{1}{C_2} \int v_{12} dt = \frac{1}{C_2} (C_1 + C_2) i_{12}$   
 $i_2 = \frac{C_2}{C_1 + C_2} i_{12}$



Next Quiz

Find  $C_{eq}$

Find  $V_K$  Kth capacitor

